Midterm Examination
Open Book, Open Notes, Time Allowed: 2.5 hours
(Sunday, May 11, 2008; 5:30pm – 8:00pm)

“I certify that I have neither received nor given unpermitted aid on this examination and
that I have reported all such incidents observed by me in which unpermitted aid is given.”

Signature __________________________

Name _______________________________  Student ID _______________________________

Problem 1 ______________ [20]
Problem 2 ______________ [18]
Problem 3 ______________ [20]
Problem 4 ______________ [38]
Problem 5 ______________ [24]
Problem 6 ______________ [20]

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TOTAL _______________ [140]
Problem 1: [20 pts] Maximum Values of Quantities. Let $X$ and $Y$ are two discrete random variables. Suppose $X$ has $m$ possible values and $Y$ has $n$ possible values.

(a) [4 pts] What is the maximum possible value of $H(X)$? What probability mass function achieves this value?

(b) [4 pts] What is the maximum possible value of $H(Y)$? What probability mass function achieves this value?

(c) [6 pts] What is the maximum possible value of $H(X, Y)$? What conditions on $X$ and $Y$ achieve this value?

(d) [6 pts] What is the maximum possible value of $I(X, Y)$? What conditions on $X$ and $Y$ achieve this value?

Problem 2: [18 pts] Binary Codes. Consider the following three sets of codeword lengths for binary variable length source codes:

$L_1 = \{2, 3, 3, 4, 5\}$
$L_2 = \{1, 2, 3, 4, 5, 5\}$
$L_3 = \{2, 2, 3, 3, 4, 4, 4, 5\}$

For each set, answer the following questions. Justify your answers.

(a) [6 pts] Is there a uniquely decodable code for this set of codeword lengths?

(b) [6 pts] Is there a prefix code for this set of codeword lengths?

(c) [6 pts] Is there an optimal code with these codeword lengths for any source (optimality is defined in terms of average codeword length). If your answer is YES, give an example of such a source (i.e., provide the probability mass function of the source).
Problem 3: [20 pts] Adder Tree. You wish to add six numbers $a_1, a_2, \ldots, a_6$ to produce the result $S = \sum_{i=1}^{6} a_i$ at the “earliest” possible time using 2-input adders each with delay of two seconds. Each 2-input adder has two inputs and produces the sum exactly two seconds after both the inputs become available.

As an example, if the numbers are available at the same time $t = 0$, then one possible adder tree that achieves your goal of producing $S$ at the earliest possible time is given below.

![Adder Tree Diagram]

Figure 1: An optimal adder tree when all inputs are available at $t = 0$

Note that the above tree produces $S$ at $t = 6$. Now suppose that the various $a_i$’s are not available at the same time but arrive at time 1, 4, 3, 5, 3, 4 seconds respectively. Try to find a tree of the 2-input adders which produces $S$ at the earliest possible time.

(a) [7 pts] What is the earliest possible time $S$ is available?

(b) [13 pts] Justify your answer.
Problem 4: [38 pts] Entropy and Codes. Consider a discrete memoryless source with an alphabet of 8 symbols has the following symbol probabilities:
\{1/36, 2/36, 3/36, 4/36, 5/36, 6/36, 7/36, 8/36\}

(a) [3 pts] Find the entropy of the source.

(b) [3 pts] What are the codeword lengths (not necessarily integers) of the optimal code?

(c) [3 pts] If Shannon code is used for this source, what will be the average codeword length?

(d) [6 pts] Find a binary Huffman code and its average codeword length.

(e) [6 pts] Find a ternary (3-valued) Huffman code and its average codeword length.

(f) [6 pts] Find the binary Elias code and its average codeword length.

(g) [6 pts] Find the binary Shannon-Fano-Elias code and its average codeword length.

(h) [5 pts] Find a bound on the probability that some uniquely decodable code will beat the optimal code found in part (b). If your answer is non-zero, how come the optimal code can be beaten?

Problem 5: [24 pts] Markov Chain. In each of the following parts, provide one example of a Markov chain, if one exists. If an example does not exist, provide the reason:

(a) [4 pts] \(H(X_1) = 1\).

(b) [5 pts] \(H(X_1) = 1\) and the entropy rate \(H(\mathcal{X}) = 1\)

(c) [4 pts] \(H(X_1) = 2\).

(d) [6 pts] \(H(X_1) = 2\) and the entropy rate \(H(\mathcal{X}) = 1\).

(e) [5 pts] \(H(X_1) = 2\) and the entropy rate \(H(\mathcal{X}) = 2\)

Problem 6: [20 pts] Entropy Rate. Consider a second order Markov process over alphabet \(\{0, 1\}\). Note that in a second order Markov process, \(X_n\) only depends on the two previous values \(X_{n-1}\) and \(X_{n-2}\) (compare it with a Markov Chain in which \(X_n\) only depends on \(X_{n-1}\)). Specifically, assume that the process we are considering is realized as follows:

\[
X_n = \begin{cases} 
    Br(0.5) & X_{n-1} = X_{n-2} \\
    Br(0.9) & \text{otherwise}
\end{cases}
\]

Find the entropy rate \(H(\mathcal{X})\)?

Hint: Define \(Z_n = (X_n, X_{n+1})\) and argue that \(\{Z_n\}\) makes a first order Markov process (a Markov Chain). Then find the entropy rate \(H(Z)\) and relate it to \(H(\mathcal{X})\).