Finite Difference Methods for Option Pricing

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Outline

Introduction

Finite Difference Methods for Option Valuation

Computer Arithmetic

Pricing American Options
Basic Definitions

Derivative.
A financial quantity that is derived based upon the value from the behavior of the underlying asset is called a Derivative.

Examples: Options, swaps, futures and forward contracts.

Option pricing. Determining the future market value of these sorts of contracts is a problem in option pricing.

Option. Option is a derivative financial security whose value depends on the value of underlying asset. Options are financial contracts that give the holder certain rights. As a holder you buy the rights stipulated in the contract. It can be the right to buy, sell or exchange one thing for another. It can be converted to cash at the expense of counterparty that issued the option, called the writer of the option. Who charges a fee for the risk of incurring possible loss?
Origin of Option: Ancient Greece

As recorded by Aristotle in Politics the fifth century BC philosopher Thales of Miletus took part in a sophisticated trading strategy. Reason of this trade was to confirm that philosophers could become rich if they so chose. This is perhaps the first rejoinder to the famous question If you are so smart, why aren’t you rich? which has dogged academics throughout the ages. Thales observed that the weather was very favorable to a good olive crop, which would result in a bumper harvest of olives. Thales put a deposit on all the olive presses surrounding Miletus. When the olive crop was harvested demand for olive presses reached enormous proportions. Thales then sublet the presses for a profit. Note that by placing a deposit on the presses, Thales was actually manufacturing an option on the olive crop that is the most he could lose was his deposit. If he had sold short olive future, he would have been liable to an unlimited loss, in the event that the olive crop turned out bad and the price of the olive went up (a surplus of olives would cause the price of olive to go down, so there were risks involved). In other words, he had the option on a future of a non storable commodity.
Basic Definitions

- Options is not an obligation

Options are exercised only when their value is greater than zero—then it is said that the option is in-the-money. All options have expiration time after that they become worthless. The payoff function describes the value of the option as a function of the underlying asset at the time of expiry. A payoff function need not be differentiable nor even continuous.
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Basic Definitions

Types of options

- Some times there are limitations on options when they can be exercised.

European option: can be exercised only at expiry.

American option: can be exercised at any time before the expiry.

Bermudan option: can be exercised at dates specified in advance.

The most basic options are the call option and the put option.

A call option is an option to buy an asset at a prescribed price $K$ (the exercise or strike price).

A put option is an option to sell an asset at a prescribed price $K$ (the exercise or strike price).
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**Figure:** An Example of European Call Option with Strike price $21. Stock price is higher than the strike price, so one can buy at the strike price and earn profit.
Exchanges Trading Options

Chicago Board Options Exchange

International Securities Exchange

NYSE Euronext

Eurex (Europe)

and many more
**Stochastic Process:** A variable whose value is changing randomly is said to follow a stochastic process.

**Geometric Brownian Motion:**
A non-dividend paying asset, $S$, following GBM is governed by SDE

$$dS = \mu S dt + \sigma S dz,$$

where $\mu$ and $\sigma$ are constants and $dS$ is the change in the level of the asset price over a small time interval $dt$.

$$\frac{dS}{S} = \mu dt + \sigma dz$$
M. S. Scholes and R. C. Merton were awarded by the Prize of the Swedish Bank for Economics in the memory of A. Nobel in 1997. Fisher Black died in 1995, was mentioned as a contributor by Swedish Academy.
The **Black-Scholes** Partial Differential Equation for Valuation of Options

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad S > 0, \quad t \in [0, T]
\]
The Black-Scholes Model for Pricing Financial Derivative

\[
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0.
\]

where \(\sigma\) is the volatility, \(r\) risk free interest rate and \(V(S, t)\) is option value at time \(t\) and stock price \(S\). Initial condition is the terminal payoff value

\[
V(S, T) = \begin{cases} 
\max\{S - X, 0\}, & \text{for call option;} \\
\max\{X - S, 0\}, & \text{for put option.}
\end{cases}
\]

Where \(T\) is the time of maturity and \(X\) is strike price.
The Transformed Black-Scholes Equation

\[ S = E \exp(x), \quad t = T - \frac{\tau}{\frac{1}{2}\sigma}, \quad C = E v(x, \tau) \]
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- $k = \frac{r}{\frac{1}{2}\sigma}$
- \[\frac{\partial v}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (k - 1)\frac{\partial v}{\partial \tau} - kv\]
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\[ v = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} u(x, \tau) \]
The Transformed Black-Scholes Equation

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\begin{align*}
\text{\textbullet} \quad S &= E \exp(x), \quad t = T - \frac{\tau}{2\sigma}, \quad C = Ev(x, \tau) \\
\text{\textbullet} \quad k &= \frac{r}{2\sigma} \\
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\text{\textbullet} \quad \frac{\partial u}{\partial \tau} &= \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad \tau > 0
\end{align*}
\]
The Transformed Black-Scholes Equation

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- Initial condition for call: \( u(x, 0) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0) \)
The Transformed Black-Scholes Equation

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- Initial condition for call: \( u(x, 0) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0) \)
- Initial condition for put: \( u(x, 0) = \max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0) \)
Although this may seem a paradox, all exact science is dominated by the idea of approximation.

Bertrand Russell
Methods to solve $u_t + Lu = 0$

- Finite difference methods
Methods to solve $u_t + Lu = 0$

- Finite difference methods
- Method of lines
Methods to solve \( u_t + Lu = 0 \)

- Finite difference methods
- Method of lines
- Collocation methods
Methods to solve $u_t + Lu = 0$

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- Finite element methods
Methods to solve $u_t + Lu = 0$

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- Method of lines
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- Monte Carlo methods
Finite Difference Method

- Numerical solution is an approximation of solution
Finite Difference Method

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- It is discrete
Finite Difference Method

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- Discretize the domain
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- Discretize the PDE/ODE
Finite Difference Method

- Numerical solution is an approximation of solution
- It is discrete
- Discretize the domain
- Discretize the PDE/ODE
- Solve the linear/nonlinear system
Synergy of FD methods

Continuous PDE $U(x,t)$ → FD methods → Discrete Difference Eqns → Solution method → $U_i^j$
Convergence of FD methods

- Accuracy
Convergence of FD methods

- Accuracy
- Consistency
Convergence of FD methods

- Accuracy
- Consistency
- Stability
Most of the numerical methods approximate the analytical solution. Often numerical value of the solution is guessed and then using iterative process that guess is refined. To perform these iterative process we use computers. As computers are faster than humans and are not susceptible to human errors, such as dropping a decimal point or miscopying a number. Computerized numerical methods provides us convenient techniques that are needed. But convenience comes with the price i.e. the introduction of error into calculations!
Types of Errors and their Sources
Types of Errors and their Sources

► Modelling Error

► Discretization and truncation error
One of the important steps in numerical computation is converting the continuous system into discrete one. This conversion process introduces the error known as discretization error. Other technique involve the truncation of the infinite series giving rise to truncation error.

► Round off and data error
Unlike the discretization and truncation error which arise due to the formulation of the numerical method, round off error and the data errors are due to the limitations of the hardware. As soon as we use the computer there are roundoff errors even we haven’t done any computation at all.
Types of Errors and their Sources
Taylor’s Theorem:

Suppose \( f(x) \) has \( (n + 1) \) derivatives in an interval containing the points \( x_0 \) and \( x_0 + h \). Then

\[
f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \ldots + \frac{h^n}{n!}f^{(n)}(x_0) + \frac{h^{n+1}}{(n + 1)!}f^{(n+1)}(\xi)
\]

where \( \xi \) is some point between \( x_0 \) and \( x_0 + h \).

OR

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - x_0)^{n+1}
\]

where \( \xi \) is some point between \( x_0 \) and \( x \).
Taylor’s Theorem:

\[ f(x) = P_n(x) + R_n(x) \], where

\[ P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \],

is called Taylor’s polynomial of degree \( n \) and

\[ R_n(x) = \frac{f^{(k+1)}(\xi_x)}{(k + 1)!} (x - x_0)^{k+1} \]

is the Remainder (truncation error).
Example:

\[ f(x) = \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k + 1)!} \]

\[ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots \]

\[ P_3(x) \]

\[ P_5(x) \]
Approximation of derivative of a function at a point $x_0$

$$ f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} $$
Example 1:

\[ \left| \frac{f(x + h) - f(x)}{h} - f'(x) \right| \leq \frac{M}{2} h = ch \]

\[ \therefore f'(x) = \frac{f(x + h) - f(x)}{h} + \mathcal{O}(h) \]
Example 2:
Using the central difference (CD) formula.

\[ \left| \frac{f(x + h) - f(x - h)}{2h} - f'(x) \right| \leq ch^2. \]

\[ f'(x) = \frac{f(x + h) - f(x - h)}{2h} + O(h^2). \]

\[ \frac{f(x + h) - f(x - h)}{2h} \rightarrow f'(x), \]

with the rate of convergence \( O(h^2) \).
Consistent: A method is consistent if its local truncation error $T_{i,j} \to 0$ as $\Delta x \to 0$ and $\Delta t \to 0$. Local truncation error is the error that occurs when the exact solution $U(x_i, t_j)$ is substituted into the FD approximation at each point of interest.
Approximation of derivative of a function at a point $x_0$

Approximate the derivative using a difference formula, instead of taking $h$ zero, take “small” values of $h$. 

### Example

- For $f(x) = \sin(x)$, $x_0 = 1.2$, 
  
  $$
  f(x_0) = \cos(1.2),
  $$
  
  $$
  \cos(x_0) \approx \sin(x_0 + h) - f(x_0)h
  $$

  Let's make a table of values of difference quotient with decreasing $h$ values.

  We hope that with decreasing $h$ the error will become smaller and smaller.
Approximation of derivative of a function at a point $x_0$

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- $f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$

Consider $f(x) = \sin(x)$, $x_0 = 1.2$.

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Approximating the derivative using a difference formula

\[ f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} \]
Approximating the derivative using a difference formula

\[ \text{Error} \leq \frac{\epsilon}{h} + \frac{M}{6}h^2 \]

where \( M = \max_{[x_0-h, x_0+h]} |f'''(x)| \)
Approximating the derivative using a difference formula

Error \leq \frac{\epsilon}{h} + \frac{M}{6}h^2

where \( M = \max_{[x_0-h,x_0+h]} |f'''(x)| \)

Optimal value of \( h \) is given by \( h = \left( \frac{3\epsilon}{M} \right)^{1/3} \)
Approximating the derivative using a difference formula

\[ \text{Error} \leq \frac{\epsilon}{h} + \frac{M}{6} h^2 \]

where \( M = \max_{[x_0-h, x_0+h]} |f'''(x)| \)

- Optimal value of \( h \) is given by \( h = \left( \frac{3\epsilon}{M} \right)^{1/3} \)
- Corresponding error is \( O(\epsilon^{2/3}) \)
Approximating the derivative using a difference formula

Round-off error

Truncation error

$h$
For a successful and acceptable approximation the approximation error dominates the roundoff error in magnitude.
First Derivative Formulas

\[ f^{(1)}(a) = \frac{-f(a - h) + f(a + h)}{2h} + \mathcal{O}(h^2) \]
First Derivative Formulas

▶ $f^{(1)}(a) = \frac{-f(a-h) + f(a+h)}{2h} + \mathcal{O}(h^2)$

▶ $f^{(1)}(a) = \frac{-3f(a) + 4f(a + h) - f(a + 2h)}{2h} + \mathcal{O}(h^2)$
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\[ f^{(1)}(a) = \frac{-2f(a - h) - 3f(a) + 6f(a + h) - f(a + 2h)}{6h} + O(h^3) \]
First Derivative Formulas

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\[ f^{(1)}(a) = \frac{f(a - 2h) - 8f(a - h) + 8f(a + h) - f(a + 2h)}{12h} + \mathcal{O}(h^4) \]
First Derivative Formulas

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\[ f^{(1)}(a) = \frac{-3f(a - h) - 10f(a) + 18f(a + h) - 6f(a + 2h) + f(a + 3h)}{12h} + \mathcal{O}(h^4) \]
First Derivative Formulas

\[ f^{(1)}(a) = \frac{-f(a-h) + f(a+h)}{2h} + \mathcal{O}(h^2) \]

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\[ f^{(1)}(a) = \frac{-11f(a) + 18f(a+h) - 9f(a+2h) + 2f(a+3h)}{6h} + \mathcal{O}(h^3) \]

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\[ f^{(1)}(a) = \frac{-25f(a) + 48f(a+h) - 36f(a+2h) + 16f(a+3h) - 3f(a+4h)}{12h} + \mathcal{O}(h^4) \]
Second Derivative Formulas

\[ f^{(2)}(a) = \frac{f(a - h) - 2f(a) + f(a + h)}{h^2} + O(h^2) \]
Second Derivative Formulas

\[ f^{(2)}(a) = \frac{f(a - h) - 2f(a) + f(a + h)}{h^2} + \mathcal{O}(h^2) \]

\[ f^{(2)}(a) = \frac{2f(a) - 5f(a + h) + 4f(a + 2h) - f(a + 3h)}{h^2} + \mathcal{O}(h^2) \]
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\[ f^{(2)}(a) = \frac{-f(a - 2h) + 16f(a - h) - 30f(a) + 16f(a + h) - f(a + 2h)}{12h^2} + O(h^4) \]
Second Derivative Formulas

\[ f^{(2)}(a) = \frac{f(a-h) - 2f(a) + f(a+h)}{h^2} + \mathcal{O}(h^2) \]

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\[ f^{(2)}(a) = \frac{11f(a-h) - 20f(a) + 6f(a+h) + 4f(a+2h) - f(a+3h)}{12h^2} + \mathcal{O}(h^3) \]
Second Derivative Formulas

\[
\begin{align*}
\Rightarrow f^{(2)}(a) &= \frac{f(a - h) - 2f(a) + f(a + h)}{h^2} + \mathcal{O}(h^2) \\
\Rightarrow f^{(2)}(a) &= \frac{2f(a) - 5f(a + h) + 4f(a + 2h) - f(a + 3h)}{h^2} + \mathcal{O}(h^2) \\
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\Rightarrow f^{(2)}(a) &= \frac{11f(a - h) - 20f(a) + 6f(a + h) + 4f(a + 2h) - f(a + 3h)}{12h^2} + \mathcal{O}(h^3) \\
\Rightarrow f^{(2)}(a) &= \frac{35f(a) - 104f(a + h) + 114f(a + 2h) - 56f(a + 3h) + 11f(a + 4h)}{h^2} + \mathcal{O}(h^3)
\end{align*}
\]
Weights and Coefficients of First Derivatives

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## Weights and Coefficients of Second Derivatives

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<td>$\frac{-1}{12}$ $\frac{4}{3}$ $\frac{-5}{2}$ $\frac{4}{3}$ $\frac{-1}{12}$</td>
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<td>6</td>
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<td>$\frac{-1}{560}$ $\frac{8}{315}$ $\frac{-1}{5}$ $\frac{8}{5}$ $\frac{-205}{72}$ $\frac{8}{5}$ $\frac{-1}{5}$ $\frac{8}{315}$ $\frac{-1}{560}$</td>
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<tr>
<td>...</td>
<td>$\cdots$ $\frac{-\frac{2}{4^2}}{\frac{2}{3^2}}$ $\frac{-\frac{2}{2^2}}{\frac{2}{1^2}}$ $\frac{-\frac{\pi^2}{3}}{\frac{2}{1^2}}$ $\frac{-\frac{2}{2^2}}{\frac{2}{3^2}}$ $\frac{-\frac{2}{4^2}}{\cdots}$</td>
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Third and Fourth Derivative Formulas

\[ f^{(3)}(a) = \frac{-f(a - 2h) + 2f(a - h) - 2f(a + h) + f(a + 2h)}{2h^3} + O(h^2) \]
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\[ f^{(3)}(a) = \frac{-5f(a) + 18f(a + h) - 24f(a + 2h) + 14f(a + 3h) - 3f(a + 4h)}{2h^3} + O(h^2) \]
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\[ f^{(4)}(a) = \frac{f(a - 2h) - 4f(a - h) + 6f(a) - 4f(a + h) + f(a + 2h)}{h^4} + O(h^2) \]
Convergence: An approximation is said to be convergent if the approximate values converge to exact values as $\Delta t \to 0$ and $\Delta x \to 0$, mathematically

$$u_h^k \to u(x_i, t_k), \quad \text{as } \Delta t \to 0 \text{ and } \Delta x \to 0$$
Convergence of finite difference methods

- The Lax Equivalence Theorem

- Consistency: A finite difference scheme is consistent if the local truncation error $\tau_{ij}$ approaches zero as $\Delta t \rightarrow 0$ and $\Delta x \rightarrow 0$ (in other words, as the mesh size approaches zero). Truncation error is the amount by which a finite difference scheme fails to satisfy the PDE.

- Stability: A method is stable if error at the initial step does not grow with iteration.
Convergence of finite difference methods

- **The Lax Equivalence Theorem**
  For a well-posed linear IVP a consistent FD scheme is convergent iff it is stable.
The Lax Equivalence Theorem

For a well-posed linear IVP a consistent FD scheme is convergent iff it is stable

Consistency: A FD scheme is consistent if the local truncation error $\tau_i^j \to 0$ as $\Delta t \to 0$ and $\Delta x \to 0$ (in other words as the mesh size approaches to zero). Truncation error is the amount by which a finite difference scheme fails to satisfy the PDE.
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- **Stability**: A method is stable if error at the initial step does not grow with iteration.
Example: FTCS Scheme

For

\[ u_t = ku_{xx} \]

the Forward-Time-Center-Space (FTCS) scheme has the truncation error of

\[ \tau = \frac{\Delta t \partial^2 u}{2 \partial t^2} + O(\Delta x)^2 \]

\[ \tau \to 0 \text{ as } \Delta t \to 0, \Delta x \to 0. \]

Hence the FTCS scheme is consistent with PDE

\[ u_t = ku_{xx}. \]
Example: Dufort-Frankel Scheme

\[ u_t = ku_{xx} \]

the Dufort-Frankel Scheme

\[
\frac{u_i^{k+1} - u_i^{k-1}}{2\Delta t} = k \frac{(u_{i+1}^{k} + u_{i-1}^{k}) - (u_{i+1}^{k+1} + u_{i-1}^{k-1})}{\Delta x^2}
\]

has the truncation error:

\[
\tau = k \frac{(\Delta x)^2}{12} \frac{\partial^4 u}{\partial x^4} - k \left( \frac{\Delta t}{\Delta x} \right)^2 \frac{\partial^2 u}{\partial t^2} - \frac{(\Delta t)^2}{6} \frac{\partial^3 u}{\partial t^3}.
\]

If \( \lim_{\Delta x \to 0} \frac{\Delta t}{\Delta x} = 0 \) then scheme is consistent.
Example: Dufort-Frankel Scheme

\[ u_t = ku_{xx} \]

the Dufort-Frankel Scheme

\[ \frac{u_i^{k+1} - u_i^{k-1}}{2\Delta t} = k \frac{(u_{i+1}^k + u_{i-1}^k) - (u_{i+1}^{k+1} + u_{i-1}^{k-1})}{\Delta x^2} \]

If \( \lim_{\Delta x \to 0} \frac{\Delta t}{\Delta x} = \beta \neq 0 \), then \( \lim_{\Delta x \to 0} \frac{\Delta t}{\Delta x} \neq 0 \)
Stability of finite difference methods

- Matrix stability analysis
Stability of finite difference methods

- Matrix stability analysis
- **von Neumann Stability Analysis:**

  $u_n^m = g^n e^{im\theta}$ in FD scheme.

  A numerical scheme for an evolution equation is stable if and only if the associated largest amplification factor satisfies $|g| = 1 + O(\Delta t)$.
Stability of finite difference methods

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- von Neumann Stability Analysis:

  Based upon Fourier analysis.
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  \]
Finite Difference Method

- Explicit Euler, Implicit Euler, and the Crank-Nicolson method.
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- Crank-Nicolson exhibits the greatest accuracy of the three for a given domain discretization.
- Finite Difference methods can be applied to American (early exercise)
Stencil for Explicit Finite Difference Scheme
Discretization of BS-PDE using the Explicit Euler Method.

\[
\frac{V^j_i - V^{j-1}_i}{\Delta t} + \frac{1}{2} \sigma^2 (i \Delta S)^2 \frac{V^j_{i+1} - 2V^j_i + V^j_{i-1}}{\Delta S^2} + r(i \Delta S) \frac{V^j_{i+1} - V^j_{i-1}}{2 \Delta S} - rV^j_i = 0
\]

\[
V^{j-1}_i = A_i V^j_{i-1} + B_i V^j_i + C_i V^j_{i+1}
\]

where

\[
A_i = \frac{1}{2} \Delta t (\sigma^2 i^2 - ri), \quad B_i = 1 - (\sigma^2 i^2 + r) \Delta t, \quad C_i = \frac{1}{2} \Delta t (ri + \sigma^2 i^2)
\]
Value of European Call Option using the Explicit Euler Method.

Figure: Solution of the Black-Scholes equation using Explicit Euler Method for European Call option, for $K = 10$, $r = 0.2$, $\sigma = 0.25$ and $T = 1$
Value of European Call Option using the Explicit Euler Method.

Figure: Solution of the Black-Scholes equation using Explicit Euler Method for European Call option, for $K = 10, r = 0.2, \sigma = 0.25$ at $T = 0$, $T/2$ and at expiry
Stencil for Implicit Finite Difference Scheme

\[ V_{i+1,j-1} \]
\[ V_{i,j-1} \]
\[ V_{i-1,j-1} \]
Value of European Put using the Implicit Euler Method.

- Mesh: 0, $\Delta S$, $2\Delta S$, $\ldots$, $M\Delta S$ where $\Delta S = S_{max}/M$
- 0, $\Delta t$, $2\Delta t$, $\ldots$, $N\Delta t$ where $\Delta t = T/N$

where $S_{max}$ is the maximum value of $S$ chosen sufficiently large

and and $V^j_i = V(i\Delta S, j\Delta t)$, $i = 0, 1, \ldots, M$, $j = 0, 1, \ldots, N$
Value of European Put using the Implicit Euler Method.

- Mesh: $0, \Delta S, 2\Delta S, \ldots, M\Delta S$ where $\Delta S = S_{\text{max}}/M$
- $0, \Delta t, 2\Delta t, \ldots, N\Delta t$ where $\Delta t = T/N$

where $S_{\text{max}}$ is the maximum value of $S$ chosen sufficiently large and and

$$V_i^j = V(i\Delta S, j\Delta t), \quad i = 0, 1, \ldots, M, \quad j = 0, 1, \ldots, N$$

- The initial and boundary conditions for the European Put are:

$$V(S, T) = \max(K - S, 0), \quad V(0, t) = Ke^{-r(T-t)}, \quad V(S_{\text{max}}, t) = 0$$

Discretized BCs are:

$$V_i^N = \max(K - (i\Delta S), 0), \quad i = 0, 1, \ldots, M$$

$$V_0^j = Ke^{-r(N-j)\Delta t}, \quad j = 0, 1, \ldots, N$$

$$V_M^j = 0, \quad j = 0, 1, \ldots, N$$
Value of European Put using the Implicit Euler Method.

- Mesh: $0, \Delta S, 2\Delta S, \ldots, M\Delta S$ where $\Delta S = S_{\text{max}}/M$
  $0, \Delta t, 2\Delta t, \ldots, N\Delta t$ where $\Delta t = T/N$

where $S_{\text{max}}$ is the maximum value of $S$ chosen sufficiently large
and and $V_i^j = V(i\Delta S, j\Delta t), i = 0, 1, \ldots, M, \quad j = 0, 1, \ldots, N$

- The initial and boundary conditions for the European Put are:

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Discretized BCs are:

\[ V_i^N = \max(K - (i\Delta S), 0), \quad i = 0, 1, \ldots, M \]
\[ V_0^j = Ke^{-r(N-j)\Delta t}, \quad j = 0, 1, \ldots, N \]
\[ V_M^j = 0, \quad j = 0, 1, \ldots, N \]

- Since we are given the payoff at expiry, our problem is to solve the Black-Scholes PDE backwards in time from expiry to the present time ($t = 0$).
Discretization of PDE using the Implicit Euler Method.

\[
\frac{V_i^j - V_i^{j-1}}{\Delta t} + \frac{1}{2} \sigma^2 (i\Delta S)^2 \frac{V_{i+1}^{j-1} - 2V_i^{j-1} + V_{i-1}^{j-1}}{\Delta S^2} + r(i\Delta S) \frac{V_{i+1}^{j-1} - V_{i-1}^{j-1}}{2\Delta S} - rV_i^{j-1} = 0
\]

\[
V_i^j = A_i V_{i-1}^{j-1} + B_i V_i^{j-1} + C_i V_{i+1}^{j-1}
\]

where

\[
A_i = \frac{1}{2} \Delta t (ri - \sigma^2 i^2), \quad B_i = 1 + (\sigma^2 i^2 + r) \Delta t, \quad C_i = -\frac{1}{2} \Delta t (ri + \sigma^2 i^2)
\]

We will use the Backslash operator to invert the tridiagonal matrix at each time step. Results for the values

\[
K = 50, r = 0.05, \sigma = 0.2, T = 3.
\]

are given in the figure.
Figure: Solution of the Black-Scholes equation using Implicit Euler Method for European Put option, for $K = 50$, $r = 0.05$, $\sigma = 0.2$ and $T = 3$
Boundary Conditions for Options

The boundary conditions for a European call are given by

\[ C(S, T) = \max(S - E, 0); S > 0 \]
\[ C(0, t) = 0; t > 0 \]
\[ C(S, t) \sim E e^{-r(T-t)} \text{ as } S \to \infty; t > 0 \]
Boundary Conditions for Options

The boundary conditions for the European put are

\[ P(S, T) = \max(E - S, 0); \ S > 0 \]
\[ P(0, t) = Ee^{-r(T-t)}; \ t > 0 \]
\[ P(S, t) \to 0 \ \text{as} \ S \to 1; \ t > 0 \]
Let $\Omega$ denote the interior of the grid and $\partial \Omega$ the boundary points.
Let $\Omega$ denote the interior of the grid and $\partial \Omega$ the boundary points.

We apply $\theta$ weighted method to discretize the PDE, where $\theta \in [0, 1]$. This is a generalization of three methods, namely, explicit, implicit and Crank-Nicolson method.

\[
\theta (u_{\Omega}^{i+1} - u_{\Omega}^{i} + Au_{\Omega}^{i} + Bu_{\partial \Omega}^{i}) + (1 - \theta) (u_{\Omega}^{i+1} - u_{\Omega}^{i} + Au_{\Omega}^{i+1} + Bu_{\partial \Omega}^{i+1}) = 0
\]

\[
(I - \theta A)u_{\Omega}^{i} = (I + (1 - \theta)A)u_{\Omega}^{i+1} + \theta Bu_{\partial \Omega}^{i} + (1 - \theta)Bu_{\partial \Omega}^{i+1}
\]
## Finite Difference Methods for the Black-Scholes Eq.

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<th>Stability</th>
<th>Convergence</th>
<th>Linear system needs to be solved</th>
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<tbody>
<tr>
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<td>Conditional</td>
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<tr>
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<td>Unconditional</td>
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<tr>
<td>1</td>
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- $\theta = 0$, Explicit method,
Finite Difference Methods for the Black-Scholes Eq.

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- $\theta = 1$, Implicit method,
## Finite Difference Methods for the Black-Scholes Eq.

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<th>Stability</th>
<th>Convergence</th>
<th>Linear system needs to be solved</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Conditional</td>
<td>$O(h^2 + k)$</td>
<td>No</td>
</tr>
<tr>
<td>$1/2$</td>
<td>Unconditional</td>
<td>$O(h^2 + k^2)$</td>
<td>Yes</td>
</tr>
<tr>
<td>1</td>
<td>Unconditional</td>
<td>$O(h^2 + k)$</td>
<td>Yes</td>
</tr>
</tbody>
</table>

- $\theta = 0$, Explicit method,
- $\theta = 1$, Implicit method,
- $\theta = 1/2$, Crank-Nicolson method.
Comparison of three methods

Figure: Solution of untransformed BS equation for Put Option with parameters

\[ E = 10; \ r = 0.05; \ T = 6/12; \ \sigma = .2; \ D = 0; \ S_{\text{min}} = 0; \ S_{\text{max}} = 100; \]
American Options

- American option allows the holder to exercise the option at any point in time up to and including expiry.

When should the holder of option exercise instead of waiting for expiry?

At expiry, the payoff of a (European or American) Put is the same, hence the boundary condition at $t = T$ is:

$$P(S, T) = \max(K - S, 0)$$

At $S = 0$, as in the European case, we expect that the payoff will again be $K$, discounted in time at the risk free rate, so that:

$$P(0, t) = Ke^{-r(T-t)}$$
American Options

- American option allows the holder to exercise the option at any point in time up to and including expiry.
- Will consider the finite difference method for American Put.
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At \( S = 0 \), as in the European case, we expect that the payoff will again be \( K \), discounted in time at the risk free rate, so that \( P(0, t) = Ke^{-r(T-t)} \).
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\[
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\]

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American Options

For the boundary as $S \to \infty$, we expect that the payoff to be zero, i.e. $P(S \to \infty, t) = 0$. 

Strategy for American Put:

$P_{Am}(S, t) = \max(K - S, PEu(S, t))$

We solve for American Put using the parameter values: $K = 50$, $r = 0.05$, $\sigma = 0.25$ and $T = 3$. 

American Options

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- When is optimal to exercise?
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- Strategy for American Put:

$$PAm(S, t) = \max(K - S, PEu(S, t))$$

- We solve for American Put using the parameter values:

$$K = 50, r = 0.05, \sigma = 0.25 \text{ and } T = 3.$$
American Options

- At each time step we need to check $V_i^j = \max(K - i\delta S, V_i^j)$. 

> For explicit method it is easy, as we have just computed $V_i^j$. 

> For implicit method, this cannot be done because at each time step we need to solve the linear system. We don't know $V_i^j$ until we get to next step. 

> Use iterative solver to solve the linear system. 

> Examples: Jacobi iteration, Gauss-Seidel method or Successive Over Relaxation or SOR iteration.
American Options

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- At each time step we need to check \( V^j_i = \max(K - i\delta S, V^j_i) \).
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- For implicit method, this cannot be done because at each time step we need to solve the linear system. We don’t know \( V^j_i \) until we get to next step.
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- Examples: Jacobi iteration, Gauss-Siedel method or Successive Over Relaxation or SOR iteration.
SOR method to solve $Ax = b$

for $k = 1, 2, \cdots, k_{\text{max}}$ do

for $i = 1, 2, \cdots, n$ do

$$y_{i}^{k+1} = \frac{1}{a_{ii}} \left[ b_{i} - \sum_{j=1}^{i-1} a_{ij}x_{j}^{k+1} - \sum_{j=i+1}^{n} a_{ij}x_{j}^{k} \right]$$

$$x_{i}^{k+1} = \omega y_{i}^{k+1} + (1 - \omega) y_{i}^{k}$$

end for

end for

where $\omega$ is called the relaxation parameter.
Example: American Put with $K = 50, \ r = 0.05, \ \sigma = 0.25, \ \omega = 1.2$

<table>
<thead>
<tr>
<th>$S$</th>
<th>Value with implicit Euler+SOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>5.8547</td>
</tr>
<tr>
<td>55</td>
<td>4.2955</td>
</tr>
<tr>
<td>60</td>
<td>3.1541</td>
</tr>
<tr>
<td>65</td>
<td>2.3202</td>
</tr>
<tr>
<td>70</td>
<td>1.7119</td>
</tr>
<tr>
<td>75</td>
<td>1.2668</td>
</tr>
<tr>
<td>80</td>
<td>0.9391</td>
</tr>
</tbody>
</table>
References

References


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References


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Thanks!!!