Linear Algebra - Final Exam Questions

There are 8 questions in this paper. You are required to answer six questions for full marks. If you answer more than six, your best answers will be taken.

1. a. List the three types of elementary row operations. State the effect of each type of operation on the determinant $\det(A)$ of a square matrix.

b. Use row operations to show

$$
\frac{yz}{zx} x = x^2 \\
\frac{zx}{xy} y = y^2 \\
\frac{xy}{yz} z = z^2
$$

$$
\begin{vmatrix}
xy & x & x^2 \\
zx & y & y^2 \\
xy & z & z^2
\end{vmatrix} = \begin{vmatrix}
1 & x^2 & x^3 \\
1 & y^2 & y^3 \\
1 & z^2 & z^3
\end{vmatrix}.
$$

c. Show that $A$ is orthogonal where

$$
A = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

2. a. Solve the following linear system for all values of $\lambda$

$$
4x_1 - 2x_2 - 7x_3 = \lambda^2 - 1 \\
x_1 + x_2 - 4x_3 = \lambda^2 + 2 \\
-5x_1 + 3x_2 + 8x_3 = \lambda
$$
clearly stating the general solution for each $\lambda$.

b. Outline a method for obtaining the inverse of a $n \times n$ matrix using row operations. Use this to find the inverse of

$$
\begin{pmatrix}
1 & 2 \\
2 & 3
\end{pmatrix}.
$$

3. a. Given a square matrix $A$, define the eigenvalues and eigenvectors.

b. Suppose that $A = \begin{pmatrix}
3 & 3 \\
1 & 5
\end{pmatrix}$. Obtain a matrix $P$ such that $P^{-1}AP = D$, where $D$ is a diagonal matrix. Give the precise form of $P$ and $D$.

c. Does the matrix $A$ in b. satisfy the Cayley-Hamilton Theorem? Show your working.
4. a. Let $V$ be a vector space of all polynomial functions in the variable $x$ over the field $\mathbb{R}$. Show that the differential and integral mappings defined by

\[
D : V \rightarrow V \text{ such that } D(f) = \frac{df}{dx},
\]

\[
\mathcal{I} : V \rightarrow \mathbb{R} \text{ such that } \mathcal{I}(f) = \int_0^1 f(x) \, dx
\]

are linear.

b. Let $V$ be the vector space of $n \times n$ matrices over the field $\mathbb{F}$. $M$ is any arbitrary matrix in $V$. Let $\phi : V \rightarrow V$ be defined by $\phi(A) = AM + MA$, where $A \in V$. Show that the mapping $\phi$ is linear.

c. Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ by $f(x, y) = (x + 1, 2y, x + y)$. Show that $f$ is not a linear mapping.

5. a. Write the vector $u = (1, -2, 5) \in \mathbb{R}^3$ as a linear combination of the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$, $u_3 = (2, -1, 1)$

b. Show that the vector $v = (2, -5, 3) \in \mathbb{R}^3$ cannot be expressed as a linear combination of the vectors $v_1 = (1, -3, 2)$, $v_2 = (2, -4, -1)$, $v_3 = (1, -5, 7)$.

c. Is the set $x_1 = (1, 2, -3)$, $x_2 = (1, -3, 2)$, $x_3 = (2, -1, 5)$ in $\mathbb{R}^3$ linearly independent.

6. Consider a $n^{th}$ order linear differential operator $L$. Write down the form of $L$ and state the two basic properties?

The differential equation

\[
 bxy' + cy = 0,
\]

where $b$, $c$ are constants, is to be solved. Obtain the general solution for this equation.

Use the solution as a guide to solve the second order differential equations

(i) $x^2y'' + 7xy' + 9y = 0$

(ii) $3x^2y'' + 6xy' + y = 0$

7. a. Let $T : V \rightarrow W$ be a linear transformation. Define the terms kernel and image of $T$.

b. Consider the transformation $T(x, y) = (x + y, x + 2y, 2x + 3y)$. Obtain $\ker T$ and use this to calculate the nullity.

c. For the transformation given in b., is the vector $(2, 1, 3)$ in the image of $T$?
8. a. Consider a vector space $V$ and a subset $U \subset V$. Define the properties for $U$ to be a subspace of $V$.

b. Consider the vector space $V = \mathbb{R}^2$ over $\mathbb{R}$. Define $U = \{(x, y) \mid x, y \in [0, \infty)\}$. Is $U$ a subspace of $\mathbb{R}^2$?

Now suppose $W = \{(-a, b) \mid a, b \in [0, \infty)\} \subset V$, i.e., the second quadrant. Define a new subset of $V$ such that $S = (U + W) \subset V$. Is $S$ a subspace of $V$?

c. A set of $n$ linearly independent vectors in $\mathbb{R}^n$ forms a basis. Does the set of vectors $(2, 4, -3), (0, 1, 1), (0, 1, -1)$ form a basis for $\mathbb{R}^3$? Explain your reasons.