Sec. 9.8 Forced Response of RLC circuit

Forced response is the solution of a second-order differential equation of the form as below:
\[ \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = f(t) \]
where \( f(t) \), called forcing function, is not zero.

<table>
<thead>
<tr>
<th>Forcing Function</th>
<th>Assumed Solution ( x_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K )</td>
<td>( A )</td>
</tr>
<tr>
<td>( Kt )</td>
<td>( At + B )</td>
</tr>
<tr>
<td>( Kt^2 )</td>
<td>( At^2 + Bt + C )</td>
</tr>
<tr>
<td>( K \sin \omega t )</td>
<td>( A \sin \omega t + BC \cos \omega t )</td>
</tr>
<tr>
<td>( K e^{-at} )</td>
<td>( A e^{-at} )</td>
</tr>
</tbody>
</table>

Table 9.8-1

Read Examples 9.8-1 and 9.8-2 on pages 382-383.

Sec. 9.9 Complete Response

Example 9.9-1

Find the complete response \( V(t) \) for \( t > 0 \).

![Circuit Diagram]

Fig 9.9-1
1. **Initial Conditions:**

At \( t = 0^- \), the steady state is:

- Inductor acts as short circuit.
- Capacitor acts as open circuit.

\[
i(0^-) = \frac{10V}{4+6} = 1 \text{ A}
\]

\[
v(0^-) = 6 \times i(0^-) = 6 \text{ V}
\]

2. Write equations for Fig 9.19-1

The circuit for \( t > 0 \):

\[
v_s = 6e^{-3t}
\]

KCL at node \( a \):

\[
\frac{v - v_s}{4} + i + \frac{i}{4} \frac{dv}{dt} = 0
\]

or

\[
\frac{dv}{dt} + v + 4i = v_s
\]

or

\[
(5 + 1)v + 4i = v_s
\]
KVL around the loop comprising inductor, resistor, capacitor

\[ v - 6i - 1 \frac{di}{dt} = 0 \]

\[ v - (s+6)i' = 0 \]  \hspace{1cm} (2)

(1) and (2) can be written as

\[
\begin{bmatrix}
(s+1) & 4 \\
1 & -(s+6)
\end{bmatrix}
\begin{bmatrix}
v \\
i
\end{bmatrix}
= 
\begin{bmatrix}
v_s \\
o
\end{bmatrix}
\]

Using Cramer's Rule,

\[
v = \frac{\begin{vmatrix}
v_s & 4 \\
o & -(s+6)
\end{vmatrix}}{\begin{vmatrix}
(s+1) & 4 \\
1 & -(s+6)
\end{vmatrix}}
\]

\[
v = \frac{-(s+6)v_s}{(s+1)(s+6) - 4}
\]

\[
v' = \frac{-(s+6)v_s}{-s^2 - 7s - 6 - 4}
\]

\[
v' = \frac{(s+6)v_s}{s^2 + 7s + 10}
\]  \hspace{1cm} (3)
**NATURAL RESPONSE:**

Get the Characteristic equation for the denominator expression in (3);

\[ s^2 + 7s + 10 = 0 \]

\[ (s + 2)(s + 5) = 0 \]

\[ s_1 = -2, s_2 = -5 \]

\[ v_n = A_1 e^{-2t} + A_2 e^{-5t} \] (4)

**FORCED RESPONSE:**

From (3), the differential equation can be written as follows:

\[ (s^2 + 7s + 10) v = (s + 6) v_f \]

\[ \frac{d^2 v}{dt^2} + 7 \frac{dv}{dt} + 10 v = \frac{dv_f}{dt} + 6 v_f \]

\[ v_f = 6 e^{-2t} \]

\[ = \frac{d}{dt} (6 e^{-2t}) + 36 e^{-3t} \]

\[ = (-18 + 36) e^{-3t} \]

\[ \frac{d^2 v}{dt^2} + 7 \frac{dv}{dt} + 10 v = 18 e^{-3t} \] (5)

From Table 9.8-1:

\[ v_f = B e^{-2t} \] (6)

To find B, substitute (6) in (5),

\[ 9B - 21B + 10B = 18 \]

\[ -2B = 18, B = -9 \]

So,

\[ v_f = -9 e^{-3t} \] (7)
COMPLETE RESPONSE

\[ v(t) = v_n(t) + v_f(t) \]

\[ v_n(t) = A_1 e^{2t} + A_2 e^{-5t} - 9e^{-3t} \]

To determine \( A_1 \) and \( A_2 \), we use initial conditions for \( v_n(0) \) and \( \frac{dv_n}{dt} \bigg|_{t=0} \).

At \( t=0 \), \( v_n(0) = A_1 + A_2 - 9 \)

\[ 6 = A_1 + A_2 - 9, \quad A_1 + A_2 = 15 \]

KCL at node \( a \), \( \frac{v_n - v_i}{4} + i + \frac{1}{4} \frac{dv_n}{dt} \bigg|_{t=0} = 0 \)

At \( t=0 \), \( \frac{6 - 6}{4} + 1 + \frac{1}{4} \frac{dv_n}{dt} \bigg|_{t=0} = 0 \)

\[ \frac{dv_n}{dt} \bigg|_{t=0} = -4 \]

Differentiating Eq. 8:

\[ \frac{dv_n}{dt} \bigg|_{t=0} = -2A_1 - 5A_2 + 27 \]

\[ 2A_1 + 5A_2 = 31 \]

From 9.9-14 and 9.9-15, \( A_1 = \frac{44}{3}, \quad A_2 = \frac{1}{3} \)

Therefore

\[ v_n = \frac{44}{3} e^{2t} + \frac{1}{3} e^{-5t} - 9e^{-3t} \]

is the complete response.

\( v_n \) and \( i \) are known as state variables.

READ Sec. 9.10 and DO EXAMPLE 9.10-1/p391-393.
STEP RESPONSE OF SERIES RLC CIRCUIT.

Find complete response for \( i(t) \).

\( V_m \) is a constant.

Initial Conditions at \( t=0^- \):

\( i(0^-) = 0, \quad V(0^-) = 0 \)

After \( t=0 \):

\[ R i + L \frac{di}{dt} + v = V_m \]

\[ (Ls + R) i + v = V_m \quad (1) \]

\[ i = C \frac{dv}{dt} \]

\[ i = Cs \frac{dv}{dt} \]

\[ v + Cs \frac{dv}{dt} = 0 \quad (2) \]

From (1) and (2):

\[
\begin{bmatrix}
Ls + R & 1 \\
-1 & Cs
\end{bmatrix}
\begin{bmatrix}
i \\
v
\end{bmatrix}
= \begin{bmatrix} V_m \\
0
\end{bmatrix}
\]

\[
v = \frac{1}{Cs(Ls + R) + 1} \begin{bmatrix}
Ls + R & V_m \\
-1 & 0
\end{bmatrix}
= \frac{O + V_m}{Cs(Ls + R) + 1}
\]

\[ (s^2 + \frac{R}{L}s + \frac{1}{LC}) v = \frac{V_m}{LC} \quad (3) \]
Characteristic equation is:

\[ s^2 + \frac{R}{L} s + \frac{1}{LC} = 0 \]

\[ s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \]

\[ s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \]

where \( \alpha = \frac{R}{2L} \) \& \( \omega_0 = \sqrt{\frac{1}{LC}} \)

**Natural Response**

\[
\begin{cases}
  s^2 > \omega_0^2, & v_n = A_1 e^{s_1 t} + A_2 e^{s_2 t} \\
  s^2 < \omega_0^2, & v_n = (A_1 \cos \omega_d t + A_2 \sin \omega_d t) e^{-\alpha t} \\
  s^2 = \omega_0^2, & v_n = (A_1 + A_2 t) e^{-\alpha t}
\end{cases}
\]

SEE TABLE 9.15-2

\[ \text{p. 403} \]

**Forced Response**

From (3):

\[
\frac{d^2 v}{dt^2} + \frac{R}{L} \frac{dv}{dt} + \frac{1}{LC} v = \frac{V_m}{LC}
\]

\[ v_f = B \]

\[ -\frac{1}{LC} B = \frac{V_m}{LC} \]

\[ B = \frac{V_m}{LC} \]

**Complete Response**

\[ v = v_n + v_m \]
For a series or parallel RLC circuit, the characteristic equations are of the same form, and roots are

\[ s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \]

where \( \alpha = \frac{1}{2RC} \) for parallel RLC circuit
\( \alpha = \frac{R}{2L} \) for series RLC

and \( \omega_0 = \frac{1}{\sqrt{LC}} \) for both.

The roots of characteristic equation can be real or complex depending on \( \alpha \) and \( \omega_0 \). The root locus can be plotted in a complex \( s \)-plane as below:

**Root Locus in Complex \( s \)-Plane**