Homework 4 – Solutions
(Was Midterm Examination-I)
(Due Friday, Oct. 07, 2005 at 11:30am (before class))

“I certify that I have neither received nor given unpermitted aid on this examination and that I have reported all such incidents observed by me in which unpermitted aid is given.”

Signature __________________________

Name _________________________________
Student ID ____________________________

Problem 1 _____________ [18]
Problem 2 _____________ [10]
Problem 3 _____________ [20]
Problem 4 _____________ [12]
Problem 5 _____________ [15]
Problem 6 _____________ [12]
Problem 7 _____________ [28]
Problem 8 _____________ [25]
Problem 9 _____________ [20]

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TOTAL _________________ [160]
Problem 1: [18 points] A signal $x(t) = \text{rect}(\frac{t-1}{3})$ is input to a filter whose impulse response is $h(t) = \text{rect}(\frac{t-3}{2})$. What is the output $y(t)$ of the filter? **Make sure to sketch and label the output.**

Solution: Assuming that the filter represents an LTI system, the output $y(t)$ is given by $y(t) = x(t) * h(t)$. The input $x(t)$ and the impulse response of the filter $h(t)$ are shown below:

![Figure 1: Impulse response and Input signal for Problem 1](image1)

Using the flip-and-slide mechanism for performing the convolution, we get the output, shown and described below:

$$y(t) = \begin{cases} 
0 & t < 1.5 \\
\text{linear increase from 0 to 2} & 1.5 < t < 3.5 \\
2 & 3.5 < t < 4.5 \\
\text{linear decrease from 2 to 0} & 4.5 < t < 6.5 \\
0 & t > 6.5 
\end{cases}$$

![Figure 2: Output signal for Problem 1](image2)
Problem 2: [10 points] Evaluate $\int_{-\infty}^{\infty} [\text{sinc}(t) \star \text{sinc}(2t)]dt$, where $\star$ denotes the convolution operation.

Solution:

\[
\begin{align*}
\text{sinc}(t) & \leftrightarrow \text{rect}(f) \\
\text{sinc}(t) & \leftrightarrow \frac{1}{2} \text{rect}(\frac{f}{2})
\end{align*}
\]

Therefore,

\[
\text{sinc}(t) \star \text{sinc}(2t) \leftrightarrow \text{rect}(f) \cdot \frac{1}{2} \text{rect}(\frac{f}{2})
\]

But, \( \int_{-\infty}^{\infty} x(t)dt = X(f)|_{f=0} \), for any transform pair \( x(t) \leftrightarrow X(f) \)

Thus,

\[
\int_{-\infty}^{\infty} [\text{sinc}(t) \star \text{sinc}(2t)]dt = \text{rect}(0) \cdot \frac{1}{2} \text{rect}(\frac{0}{2})
\]

\[
\Rightarrow \int_{-\infty}^{\infty} [\text{sinc}(t) \star \text{sinc}(2t)]dt = \frac{1}{2}
\]

Obviously, the same result can also be obtained by direct integration.
Problem 3: [20 points] Consider two time domain functions, \( f(t) \) and \( g(t) \). We are interested in finding if the composite function \( f(g(t)) \) is even, odd, or neither. Determine this for the following cases (for each case, your answer should be even, odd, or neither. justify your answers; almost no credit will be given without justification. **Hint:** A signal \( h(x) \) is even if \( h(-x) = h(x) \) and is odd if \( h(-x) = -h(x) \):

(a) [5 points] \( f(t) \) is odd and \( g(t) \) is even.
\[
f(g(-t)) = f(g(t)) \quad \implies \text{EVEN}
\]

(b) [5 points] \( f(t) \) is odd and \( g(t) \) is odd.
\[
f(g(-t)) = f(-g(t)) = -f(g(t)) \quad \implies \text{ODD}
\]

(c) [5 points] \( f(t) \) is even and \( g(t) \) is even.
\[
f(g(-t)) = f(g(t)) \quad \implies \text{EVEN}
\]

(d) [5 points] \( f(t) \) is even and \( g(t) \) is odd.
\[
f(g(-t)) = f(-g(t)) = f(g(t)) \quad \implies \text{EVEN}
\]
**Problem 4: [12 points]** A raised cosine pulse is frequently used in digital communications, and is given by:

\[ r(t) = \text{rect}(\frac{t}{2})(1 + \cos \pi t) \]

Find the Fourier transform of raised cosine pulse by direct integration. Simplify your answers expressing them in terms of \( \text{sinc} \) functions.

**Solution:**

\[
R(f) = \int_{-\infty}^{\infty} r(t)e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} \left( \text{rect}(\frac{t}{2}) + \text{rect}(\frac{t}{2}) \cos \pi t \right) e^{-j2\pi ft} dt \\
= \int_{-1}^{1} (1 + \cos \pi t)e^{-j2\pi ft} dt = \int_{-1}^{1} e^{-j2\pi ft} dt + \int_{-1}^{1} \cos \pi t e^{-j2\pi ft} dt \\
= \left. \frac{e^{-j2\pi ft}}{-j2\pi f} \right|_{-1}^{1} + \int_{-1}^{1} \left( \frac{e^{j\pi t} + e^{-j\pi t}}{2} \right) e^{-j2\pi ft} dt \\
= \frac{e^{-j2\pi f} - e^{j2\pi f}}{-j2\pi f} + \frac{e^{-j\pi (2f+1)t}}{-j2\pi (2f+1)} \bigg|_{-1}^{1} + \frac{e^{-j\pi (2f-1)t}}{-j2\pi (2f-1)} \bigg|_{-1}^{1} \\
= \sin 2\pi f \cdot \frac{\pi f}{\pi f} + \frac{e^{-j\pi (2f+1) - e^{j\pi (2f+1)}}}{-j2\pi (2f+1)} + \frac{e^{-j\pi (2f-1) - e^{j\pi (2f-1)}}}{-j2\pi (2f-1)} \\
= 2 \text{sinc } 2f + \frac{\sin \pi (2f + 1)}{\pi (2f + 1)} + \frac{\sin \pi (2f - 1)}{\pi (2f - 1)} \\
= 2 \text{sinc } 2f + \text{sinc } (2f + 1) + \text{sinc } (2f - 1) 
\]

The same result is obtained by using the properties of Fourier transforms.
Problem 5: [15 points] A radar can measure the distance to a target by transmitting a waveform, receiving the reflected waveform back from the target and cross-correlating the transmitted and received waveforms. A maximum value of cross-correlation indicates the amount of delay between the transmitted and received waveforms. Let $x(t)$ is the transmitted signal shown below, and $y(t)$ is the received signal also shown below.

![Transmitted and Received Signal from a Radar](image)

Solution:

(a) [10 points] For what value of $\tau$ is $R_{xy}(\tau)$ a maximum? (Use the rest of the page and next page for working; write your final answer here).

The plot of $R_{xy}(\tau)$ is as shown in the following figure, from which we notice that the maximum value of $R_{xy}(\tau)$ occurs at $t = 10\mu s$.

![Cross-correlation between transmitted and received signals](image)

(b) [5 points] If the time units used in the figure are microseconds, how far is the target from the radar? You may assume that the electromagnetic signal propagates with the speed of light.

The delay between the transmitted and received signal is $t = 10\mu s$. Thus, the distance $d$ between transmitter and receiver is obtained from $2d = c \times t$, where $c = 3 \times 10^8$ m/s is the speed of light. This yields $d = 1500$ m.
Problem 6: [12 points] Consider two baseband signals $x_1(t)$ and $x_2(t)$ having bandwidths $B_1$ and $B_2$ Hertz, respectively. We need to transmit $y(t) = x_1(t) \cdot x_2(t)$ through a digital communication system. The transmitter must sample $y(t)$ before further processing. The goal is to correctly recover $y(t)$ at the receiver, at least when the noise can be neglected.

Solution:

(a) [6 points] Is there a minimum limit on the separation between the samples taken from $y(t)$? If yes, what is it? If not, why?

There is no minimum limit on the separation between samples, except what posed by electronics or the bandwidth of the communication medium. That is, it is okay to sample the signal at a faster than required rate from a communication theoretic point of view.

(b) [6 points] Is there a maximum limit on the separation between the samples taken from $y(t)$? If yes, what is it? If not, why?

There is a maximum limit on the separation between samples, and is dictated by the Nyquist sampling theorem. The maximum separation is $\frac{1}{2B}$, where $B$ is the maximum frequency component in $y(t)$. Since $y(t) = x_1(t) \cdot x_2(t)$, we have $Y(f) = X_1(f) \ast X_2(f)$, where $\ast$ denotes convolution. Now the convolution of two signals results in a signal whose bandwidth is the sum of individual bandwidth of convolving signals. Thus, $B = B_1 + B_2$, and, therefore, the separation between samples must be smaller than $\frac{1}{2(B_1+B_2)}$. 
Problem 7: [28 points] For each of the following systems, output \( y(t) \) is given in terms of input \( x(t) \). State whether these systems are linear or non-linear. Also state whether these systems are time invariant or time variant?

**Solution:** First we assume that \( G[.] \) denotes the application of a system (LTI or not) on its input. That is, \( y(t) = G[x(t)] \). Also assume that \( a_1 \) and \( a_2 \) are constants, in the following.

(a) [4 points] \( y(t) = x(t) \cos 2\pi f t \)

\[
G[a_1x_1(t) + a_2x_2(t)] = (a_1x_1(t) + a_2x_2(t)) \cos 2\pi ft \\
= a_1x_1(t) \cos 2\pi ft + a_2x_2(t) \cos 2\pi ft \\
= a_1G[x_1(t)] + a_2G[x_2(t)] \quad \Rightarrow \text{LINEAR}
\]

\[
G[x(t - t_0)] = x(t - t_0) \cos 2\pi ft \\
\neq x(t - t_0) \cos 2\pi f(t - t_0) = y(t - t_0) \quad \Rightarrow \text{TIME VARYING}
\]

(b) [4 points] \( y(t) = \sin[x(t)] \)

\[
G[a_1x_1(t) + a_2x_2(t)] = \sin(a_1x_1(t) + a_2x_2(t)) \\
\neq a_1 \sin x_1(t) + a_2 \sin x_2(t) \\
= a_1G[x_1(t)] + a_2G[x_2(t)] \quad \Rightarrow \text{NON LINEAR}
\]

\[
G[x(t - t_0)] = \sin[x(t - t_0)] = y(t - t_0) \quad \Rightarrow \text{TIME INVARIANT}
\]

(c) [4 points] \( y(t) = \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi ft}d\tau \)

\[
G[a_1x_1(t) + a_2x_2(t)] = \int_{-\infty}^{\infty} [a_1x_1(\tau) + a_2x_2(\tau)]e^{-j2\pi ft}d\tau \\
= a_1 \int_{-\infty}^{\infty} a_1x_1(\tau)e^{-j2\pi ft}d\tau + a_2 \int_{-\infty}^{\infty} a_2x_2(\tau)e^{-j2\pi ft}d\tau \\
= a_1G[x_1(t)] + a_2G[x_2(t)] \quad \Rightarrow \text{LINEAR}
\]

\[
G[x(t - t_0)] = \int_{-\infty}^{\infty} x(\tau - t_0)e^{-j2\pi ft}d\tau \\
= \int_{-\infty}^{\infty} x(\tau')e^{-j2\pi f(t' + t_0)}d\tau' \quad (\text{using } \tau' = t - t_0) \\
= \int_{-\infty}^{\infty} x(\tau')e^{-j2\pi f \tau'}e^{-j2\pi ft_0}d\tau' \\
= \int_{-\infty}^{\infty} x(\tau')e^{-j2\pi f(t - t_0)\tau'}e^{-j2\pi ft_0(t + \tau')}d\tau' \\
\neq \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f(t - t_0)\tau}d\tau = y(t - t_0) \quad \Rightarrow \text{TIME VARYING}
\]
(d) [4 points] \( y(t) = \frac{d}{dt} x(t) \)

\[
G[a_1x_1(t) + a_2x_2(t)] = \frac{d}{dt} (a_1x_1(t) + a_2x_2(t)) \\
= a_1 \frac{dx_1(t)}{dt} + a_2 \frac{dx_2(t)}{dt} \\
= a_1G[x_1(t)] + a_2G[x_2(t)] \quad \Rightarrow \text{LINEAR}
\]

\[
G[x(t - t_0)] = \frac{d}{dt} x(t - t_0) \\
= y(t - t_0) \quad \Rightarrow \text{TIME INVARIANT}
\]

(e) [4 points] \( y(t) = \cos[2\pi ft + x(t)] \)

\[
G[a_1x_1(t) + a_2x_2(t)] = \cos(2\pi ft + a_1x_1(t) + a_2x_2(t)) \\
\neq a_1 \cos(2\pi ft + x_1(t)) + a_2 \cos(2\pi ft + x_2(t)) \\
= a_1G[x_1(t)] + a_2G[x_2(t)] \quad \Rightarrow \text{NON LINEAR}
\]

\[
G[x(t - t_0)] = \cos[2\pi ft + x(t - t_0)] \\
\neq \cos[2\pi f(t - t_0) + x(t - t_0)] \\
= y(t - t_0) \quad \Rightarrow \text{TIME VARYING}
\]

(f) [4 points] \( y(t) = x(-t) \)

\[
G[a_1x_1(t) + a_2x_2(t)] = a_1x_1(-t) + a_2x_2(-t) \\
= a_1G[x_1(t)] + a_2G[x_2(t)] \quad \Rightarrow \text{LINEAR}
\]

\[
G[x(t - t_0)] = x(-t - t_0) \\
\neq x(-(t - t_0)) = y(t - t_0) \quad \Rightarrow \text{TIME VARYING}
\]

(g) [4 points] \( y(t) = \text{sgn}(x(t)) \)

\[
G[a_1x_1(t) + a_2x_2(t)] = \text{sgn}(a_1x_1(t) + a_2x_2(t)) \\
\neq a_1\text{sgn}(x_1(t)) + a_2\text{sgn}(x_2(t)) \\
= a_1G[x_1(t)] + a_2G[x_2(t)] \quad \Rightarrow \text{NON LINEAR}
\]

\[
G[x(t - t_0)] = \text{sgn}(t - t_0) \\
= y(t - t_0) \quad \Rightarrow \text{TIME INVARIANT}
\]
Problem 8: [25 points] Consider a signal \( x(t) \), with average value \( \langle x(t) \rangle = 0 \), and \( S_x = \langle x^2(t) \rangle = \frac{1}{3} \). The signal \( x(t) \) has a maximum value 0.8 and a minimum value -0.8. Suppose we need to generate an AM signal \( x_c(t) = A_c(1 + \mu x(t)) \cos 2\pi f_c t \) for transmission from an AM radio station, whose listeners would use standard envelope detectors.

Solution:

(a) [5 points] What is the maximum value of \( \mu \) that can be used, while ensuring that envelope detection remains possible at the receivers?

For the envelope to be preserved, the envelop must remain positive for all values of input signal. In the worst case, \( 1 + \mu \min_t x(t) \geq 0 \). Thus, \( 1 - .8\mu \geq 0 \), which implies that \( \mu \leq 1.25 \).

(b) [10 points] Assume that the value of \( \mu \) found above is used. We also note that AM transmitters put maximum value constraints on two parameters: the average transmitted power \( S_T \), and the peak transmitted power \( A_{max}^2 \), where \( A_{max} \) is the maximum absolute value the modulated signal can take. If both the constraints are to be exactly satisfied simultaneously, derive a relation between \( A_{max} \) and \( S_T \).

First \( A_{max} = A_c(1 + \mu \max_t x(t)) \)
\[ = A_c(1 + 1.25 \times 0.8) = 2A_c \]

Next \( S_T = P_c + 2P_{sb} \)
\[ = \frac{1}{2} A_c^2 + 2 \cdot \frac{1}{4} A_c^2 \mu^2 S_x \]
\[ = 0.76042A_c \quad \text{(for } \mu = 1.25 \text{ and } S_x = \frac{1}{3} \text{)} \]
\[ = 0.1901A_{max}^2 \]

(c) [10 points] Repeat part (b) if \( \mu = \frac{3}{4} \) is used.

First \( A_{max} = A_c(1 + \mu \max_t x(t)) \)
\[ = A_c(1 + \frac{3}{4} \times 0.8) = 1.6A_c \]

Next \( S_T = P_c + 2P_{sb} \)
\[ = \frac{1}{2} A_c^2 + 2 \cdot \frac{1}{4} A_c^2 \mu^2 S_x \]
\[ = 0.59375A_c \quad \text{(for } \mu = \frac{3}{4} \text{ and } S_x = \frac{1}{3} \text{)} \]
\[ = 0.2319A_{max}^2 \quad \text{(using } A_{max} = 1.6A_c \text{)} \]
Problem 9: [20 points] Define a random process \( Y(t) = \cos(2\pi f_c t + \Theta_1) \cdot \cos(2\pi f_c t + \Theta_2) \), where \( \Theta_1 \) and \( \Theta_2 \) are independent and identically distributed uniform random variables between 0 and \( 2\pi \). That is, \( \Theta_1 \sim U(0, 2\pi) \) and \( \Theta_2 \sim U(0, 2\pi) \). Is \( Y(t) \) a WSS process? Justify your answer and show the working. If yes, find its power spectral density \( S_Y(f) \)?

Solution:

\[
E[Y(t)] = E[\cos(2\pi f_c t + \Theta_1) \cos(2\pi f_c t + \Theta_2)] \\
= E[\cos(2\pi f_c t + \Theta_1)] E[\cos(2\pi f_c t + \Theta_2)] \quad (\Theta_1 \text{ and } \Theta_2 \text{ are independent})
\]

But,
\[
E[\cos(2\pi f_c t + 2\Theta_1)] = E[\cos(2\pi f_c t + 2\Theta_2)] = \frac{1}{2\pi} \int_0^{2\pi} \cos(2\pi f_c t + \theta) d\theta = 0
\]

Thus,
\[
E[Y(t)] = 0
\]

\[
R_Y(t_1, t_2) = E[\cos(2\pi f_c t_1 + \Theta_1) \cos(2\pi f_c t_2 + \Theta_2) \cos(2\pi f_c t_2 + \Theta_1) \cos(2\pi f_c t_2 + \Theta_2)] \\
R_Y(t_1, t_2) = E[\cos(2\pi f_c t_1 + \Theta_1) \cos(2\pi f_c t_2 + \Theta_1)] E[\cos(2\pi f_c t_1 + \Theta_2) \cos(2\pi f_c t_2 + \Theta_2)] \\
= \frac{1}{2} \cos(2\pi f_c (t_1 - t_2)) \times \frac{1}{2} \cdot \cos(2\pi f_c (t_1 - t_2)) \quad \text{(using results from class)} \\
= \frac{1}{4} \cos^2(2\pi f_c (t_1 - t_2))
\]

Hence \( Y(t) \) is a WSS process. To find \( S_Y(f) \), we take the Fourier transform of the autocorrelation function \( R_Y(\tau) \) as:

\[
R_Y(\tau) = \frac{1}{4} \cos^2 2\pi f_c \tau = \frac{1}{8} (1 + \cos 4\pi f_c \tau) \\
S_Y(f) = \frac{1}{8} \delta(f) + \frac{1}{16} \delta(f - 2f_c) + \frac{1}{16} \delta(f + 2f_c)
\]

Thus,
\[
S_Y(f) = \frac{1}{8} \delta(f) + \frac{1}{16} \delta(f - 2f_c) + \frac{1}{16} \delta(f + 2f_c)
\]