Final Examination – Solutions

(Open Books, Open Notes)
Time Allowed: 3 hours (Monday, November 28, 2005)

“I certify that I have neither received nor given unpermitted aid on this examination and
that I have reported all such incidents observed by me in which unpermitted aid is given.”

Signature ___________________________

SOLUTIONS

Name _____________________________

Student ID ___________________________

IMPORTANT NOTE: For your reference, a plot of Q-function is attached at the end of
this exam book. Also attached is a table which provides the area under the gaussian curve
of zero mean and unit variance, to the left of a point.

Problem 1 _____________ [21]
Problem 2 _____________ [30]
Problem 3 _____________ [24]
Problem 4 _____________ [40]
Problem 5 _____________ [35]

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TOTAL _____________ [150]
Problem 1: Digital PAM Formats or Line Codes [21 points] Consider the various digital PAM formats given in the text. For each of the format we consider, let the Digital PAM signal is given by:

$$x(t) = \sum_k a_k p(t - kD)$$

What is $a_k, p(t)$, and $P(f)$ for each of the following formats?

(a) [3 points] Unipolar NRZ.

$$a_k \in \{0, 1\}$$

$$p(t) = A \cap \left( \frac{t}{D} \right) \quad \text{(or any shifted version)}$$

$$P(f) = AD \text{sinc}(Df)$$

(b) [3 points] Unipolar RZ.

$$a_k \in \{0, 1\}$$

$$p(t) = A \cap \left( \frac{t + D}{D^2} \right) \quad \text{(or any shifted version)}$$

$$P(f) = \frac{AD}{2} \text{sinc} \left( \frac{D}{2} f \right) e^{-j2\pi f \frac{D}{4}}$$

(c) [3 points] Polar NRZ.

$$a_k \in \{-1, 1\}$$

$$p(t) = A \cap \left( \frac{t}{D} \right) \quad \text{(or any shifted version)}$$

$$P(f) = AD \text{sinc}(Df)$$

(d) [3 points] Polar RZ.

$$a_k \in \{-1, 1\}$$

$$p(t) = A \cap \left( \frac{t + D}{D^2} \right) \quad \text{(or any shifted version)}$$

$$P(f) = \frac{AD}{2} \text{sinc} \left( \frac{D}{2} f \right) e^{-j2\pi f \frac{D}{4}}$$

(e) [3 points] Bipolar NRZ.

$$a_k \in \{0, (-1)^n\}$$

$$p(t) = A \cap \left( \frac{t}{D} \right) \quad \text{(or any shifted version)}$$

$$P(f) = AD \text{sinc}(Df)$$
where \( n \) is the number of 1s observed previous to current bit. Alternately,

\[
a_k \in \{0, 1\}
\]

\[
p(t) = (-1)^n A \cap \left( \frac{t}{D} \right) \quad \text{(or any shifted version)}
\]

\[
P(f) = (-1)^n ADsinc(Df)
\]

(f) [3 points] Twinned binary (or Split-phase Manchester).

\[
a_k \in \{-1, 1\}
\]

\[
p(t) = A \cap \left( \frac{t + \frac{D}{4}}{D} \right) - A \cap \left( \frac{t - \frac{D}{4}}{D} \right) \quad \text{(or any shifted version)}
\]

\[
P(f) = \frac{AD}{2} \text{sinc} \left( \frac{D}{2} f \right) e^{-j2\pi f \frac{D}{4}} - \frac{AD}{2} \text{sinc} \left( \frac{D}{2} f \right) e^{j2\pi f \frac{D}{4}}
\]

(g) [3 points] polar quaternary NRZ.

\[
a_k \in \{-1, +1, -3, +3\} \quad \text{(one of the numerous correct solutions!)}
\]

\[
p(t) = A \cap \left( \frac{t}{D} \right) \quad \text{(or any shifted version)}
\]

\[
P(f) = ADsinc(Df)
\]
Problem 2: ML and MAP Detectors [30 points] In class, we considered the detectors for noisy binary channels, assuming that the input symbols (i.e., 0 and 1) appear with equal frequency. This is usually true but not always. Even when the two symbols appear in different proportions, the exact proportion may not be known at the receiver which has no choice other than assuming that the probability of sending a 1 is the same as the probability of sending a 0. The resulting detector which assumes 0s and 1s in equal proportions is called a Maximum Likelihood (ML) detector and was studied in class. If, however, we know the proportions of 1s and 0s being sent over the channel, we can do better by using a Bayesian detector, also called the MAP detector. The following problem compares the performance of ML and MAP detectors.

Consider a transmitter sending binary data using a signal $X$ where $X = -1$ and $X = +1$ are the possible signal values. The transmitter sends $X$ over an AWGN channel. After filtering at the receiver, the zero-mean received gaussian noise $N$ has power equal to 0.25. The received signal $Y = X + N$ is fed to a detector which makes an estimate of the transmitted symbol.

(a) [6 points] Suppose $\Pr\{X = -1\} = 0.5$. Find the error probability $P_e$ of a ML detector. Note that we studied the ML detector in class and it is the one which decides in favor of $x$ such that $f_{Y|X}(y|x)$ is maximized. For example, in the binary case at hand, ML detector will decide that +1 was sent if $f_{Y|X}(y|+1) > f_{Y|X}(y|-1)$, and a -1 was sent otherwise.

Solution: First note that the distribution of $N$ is:

$$f_N(n) = \frac{1}{\sqrt{2\pi \times 0.25}} e^{-\frac{n^2}{2\times0.25}}$$

$$= \sqrt{\frac{2}{\pi}} e^{-2n^2}$$

Next, note the conditional distributions of $Y$ as:

$$f_{Y|X}(y|1) = f_N(y-1) = \sqrt{\frac{2}{\pi}} e^{-2(y-1)^2}$$

$$f_{Y|X}(y|-1) = f_N(y+1) = \sqrt{\frac{2}{\pi}} e^{-2(y+1)^2}$$

The ML detector decides in favor of +1 if $f_{Y|X}(y|+1) > f_{Y|X}(y|-1)$. Either by equating $f_{Y|X}(y|+1)$ and $f_{Y|X}(y|-1)$ or by using symmetry, the threshold for the observed value is $y_{th} = 0$. Thus, ML detector decides in favor of +1 if the observed value $Y > 0$. The error probability is given by:

$$P_e = P_{-1}P_{e,-1} + P_{+1}P_{e,+1}$$
where $P_{-1}$ is the probability that -1 was sent, $P_{+1}$ is the probability that +1 was sent, $P_{e,-1}$ is the probability of error given that -1 was sent, and $P_{e,+1}$ is the probability of error given that +1 was sent. By symmetry, $P_{e,-1} = P_{e,+1}$ and, therefore, $P_e = P_{e,-1}$ computed as:

$$P_e = P_{e,-1} = Pr\{Y > 0|X = -1\}$$
$$= \int_0^\infty f_Y|X(y)| - 1$$
$$= \int_0^\infty \sqrt{\frac{2}{\pi}} e^{-2(y+1)^2} dy$$
$$= \int_1^\infty \sqrt{\frac{2}{\pi}} e^{-2w^2} dw$$
$$= Q\left(\frac{1}{\sqrt{2.25}}\right) = Q(2) = 0.0228$$

(b) [6 points] Suppose $Pr\{X = -1\} = 0.5$. Find the error probability $P_e$ of a MAP detector. A MAP detector maximizes the a-posteriori probability such that it decides in favor of $x$ if $f_{X|Y}(x|y)$ is maximized. (Hint: using Bayes’ rule, first show that maximizing $f_{X|Y}(x|y)$ is equivalent to maximizing $Pr\{X = x\} f_{Y|X}(y|x)$).

**Solution:** The MAP detector decides in favor of +1 if $f_{X|Y}(+1|y) > f_{X|Y}(-1|y)$. Note that, using Bayes’ rule, we can write:

$$f_{X|Y}(x|y) = \frac{f_Y|X(y|x)Pr\{X = x\}}{f_Y(y)}$$

Thus, the MAP detector maximizes $f_Y|X(y|x)Pr\{X = x\}$. For this part, $Pr\{X = -1\} = Pr\{X = +1\} = 0.5$, therefore, the MAP detector also maximizes $f_Y|X(y|x)$ which is the same as the ML detector. The the error probability is 0.0228 and the decision threshold value is $y_{th} = 0$.

(c) [5 points] Suppose $Pr\{X = -1\} = 0.25$, find the error probability $P_e$ of a ML detector.

**Solution:** Since ML detector maximizes $f_Y|X(y|x)$, its decision threshold remains the same by changing the probabilities of occurrence of input symbols. Thus, in this case, the error probabilities are:

$$P_{e,-1} = P_{e,+1} = Q(2) = 0.0228$$
$$P_e = P_{-1}P_{e,-1} + P_{+1}P_{e,+1}$$
$$= 0.25 \times 0.0228 + 0.75 \times 0.0228 = 0.0228$$

(d) [6 points] Suppose $Pr\{X = -1\} = 0.25$, find the threshold to be used by a MAP detector in deciding whether a +1 or a -1 was sent. (You can trade this answer for 6 points; it will be used in the next part. Make sure to read the next part before trading!)
Solution: From above, the MAP detector maximizes $f_{Y|X}(y|x) Pr\{X = x\}$. Thus, it decides in favor of -1 if:

$$f_{Y|X}(y| -1) Pr\{X = -1\} > f_{Y|X}(y| +1) Pr\{X = +1\}$$

$$\Rightarrow \ 0.25 \sqrt{\frac{2}{\pi}} e^{-2(y+1)^2} > 0.75 \sqrt{\frac{2}{\pi}} e^{-2(y-1)^2}$$

and decides in favor of +1, otherwise. Therefore, the threshold is found from:

$$0.25 \sqrt{\frac{2}{\pi}} e^{-2(y_{th}+1)^2} = 0.75 \sqrt{\frac{2}{\pi}} e^{-2(y_{th}-1)^2}$$

which yields, $y_{th} = -0.1373$.

(e) [7 points] Using the result from part (d), find the error probability $P_e$ of a MAP detector when $Pr\{X = -1\} = 0.25$.

Solution: Using the threshold value $y_{th}$ from above,

$$P_e = P_{-1} P_{e,-1} + P_{+1} P_{e,+1}$$

$$= 0.25Q\left(\frac{1 - 0.1373}{0.5}\right) + 0.75Q\left(\frac{1 + 0.1373}{0.5}\right)$$

$$= 0.25 \times 0.0432 + 0.75 \times 0.0115 = 0.019425$$

Thus, the MAP detector is performing better than ML detector when the input symbols have unequal probabilities. In fact, MAP is the optimum detector in terms of minimizing the probability of error.
Problem 3: Power Calculations in AM [24 points] A message signal consists of three
tones and is given by:

\[ x(t) = C(\cos 2\pi f_m t + 3 \cos 4\pi f_m t + 2 \cos 8\pi f_m t) \]

The message signal \( x(t) \) is input to an AM modulator with \( \mu = 1 \), which generates a signal
given by:

\[ x_c(t) = A_c[1 + \mu x(t)] \cos 2\pi f_c t; \quad f_c \gg f_m \]

(a) [5 points] What is the maximum value \( C \) can take such that the AM signal generated can
be correctly demodulated by an envelope detector? (Note: You must get it right. Otherwise,
you will lose points on subsequent parts also. If you are unsure, you can trade the answer
for this part from the instructor for 5 points. That is, if you trade the answer, you will get
0 on this part but you will be guaranteed to have a correct answer for subsequent parts.)

**Solution:** For demodulation by an envelope detector, \( \mu |x(t)|_{\text{max}} \leq 1 \). For \( \mu = 1 \), we have
\( |x(t)|_{\text{max}} \leq 1 \), which implies that \( C(2+1+3) \leq 1 \), or \( C \leq \frac{1}{6} \). We use the maximum value
for keeping sideband to carrier power maximized. thus, we have \( C = \frac{1}{6} \).

For the rest of the problem, value of \( C \) in part (a) is used.

(b) [4 points] What are the minimum and maximum values of the AM envelope?

**Solution:** With the value of \( C \) chosen above, the minimum value of the envelope is 0 while
the maximum value is computed by considering that \( x_c(t) = A_c[1 + x(t)] \cos 2\pi f_c t \), which
gives the maximum value of the AM envelope as \( A_c(1+1) = 2A_c \).

(c) [8 points] Find numerical value for \( \frac{P_{SB}}{P_c} \) where \( P_{SB} \) is the power in ONE sideband and \( P_c \)
is the power in the carrier. (Warning: Be extra careful in calculations. We really need the
exact answers)

**Solution:** First note that,

\[
x_c(t) = A_c[1 + \frac{1}{6}x(t)] \cos 2\pi f_c t
\]

\[
= A_c[1 + \frac{1}{6}(\cos 2\pi f_m t + 3 \cos 4\pi f_m t + 2 \cos 8\pi f_m t)] \cos 2\pi f_c t
\]

\[
= A_c \cos 2\pi f_c t + \frac{1}{6}(\cos 2\pi f_m t + 3 \cos 4\pi f_m t + 2 \cos 8\pi f_m t) \cos 2\pi f_c t
\]

Next we note that each term like \( \cos 2\pi nf_m t \cos 2\pi f_c t \) can be written as sum of two cosines:
\( \frac{1}{2} \cos 2\pi(f_c + nf_m)t \) which represents the upper sideband, and \( \frac{1}{2} \cos 2\pi(f_c - nf_m)t \) which
represents the lower sideband. The power in each of these sidebands is \( \frac{1}{4} \times \frac{1}{2} \) or \( \frac{1}{8} \). Now the
powers in carrier and a sideband are given by:

\[ P_c = \frac{A_c^2}{2} \]

\[ P_{SB} = \frac{A_c^2}{36} \left( \frac{1}{8} + \frac{9}{8} + \frac{1}{2} \right) \]

\[ = \frac{7}{144} A_c^2 \]

Thus, \[ \frac{P_{SB}}{P_c} = \frac{7}{144} \cdot \frac{1}{2} = \frac{7}{72} \]

Alternately, \[ P_{SB} = \frac{1}{4} A_c^2 S_x \mu^2 \] and \[ P_c = \frac{1}{2} A_c^2 \] which gives \[ \frac{P_{SB}}{P_c} = \frac{1}{2} S_x \]. Then, we find \( S_x \) as \[ S_x = \frac{1}{36} \left( \frac{1}{2} + \frac{9}{2} + \frac{4}{2} \right) = \frac{7}{36} \], which yields \[ \frac{P_{SB}}{P_c} = \frac{1}{2} S_x = \frac{7}{72} \].

(d) [7 points] Find numerical value for \( \frac{P_c}{S_T} \) where \( S_T \) is the total transmitted power.

**Solution:** Noting that \( P_{SB} \) is the power in one sideband, we have,

\[ S_T = P_c + 2P_{SB} \]

\[ = A_c^2 \left( \frac{1}{2} + 2 \cdot \frac{7}{144} \right) \]

\[ = \frac{43}{72} A_c^2 \]

Thus, \[ \frac{P_c}{S_T} = \frac{1}{2} \cdot \frac{43}{72} = \frac{36}{43} \]
Problem 4: Long-distance Optical Cable [40 points] An optical cable is laid between Lahore and Karachi with an approximate length of 1200 km. The cable incurs a signal power loss of 0.5 dB/km. This necessitates the use of \( m \) identical repeater sections. Each repeater (including the receiver) boosts the signal power to just compensate for the loss in the previous section. That is, the power gain at each repeater is the same as power loss in the previous section. As a result, input power to each section is the same and is equal to the power at the output of the receiver. Let \( P_{\text{in}} \) is the power input to the first section and \( P_{\text{out}} \) is the power output from the receiver. For this problem, use \( P_{\text{in}} = P_{\text{out}} = 100 \text{ mW} \). The cable is used for transmitting binary PAM data with unipolar NRZ pulses.

(a) [6 points] If the minimum input power required at each repeater (and at the receiver) is 10 \( \mu \text{W} \), what is the minimum required value of \( m \)? (Note: You must get it right. Otherwise, you will lose points on subsequent parts also. If you are unsure, you can trade the answer for this part from the instructor for 6 points. That is, if you trade the answer, you will get 0 on this part but you will be guaranteed to have a correct answer for subsequent parts. Make sure to read the subsequent parts before trading the points for this part.)

**Solution:** The transmitted power at the input and from each repeater is 100 mW while the minimum received power at each repeater and at the receiver is 10 \( \mu \text{W} \). This means that the maximum loss allowed on each segment is \( \frac{100 \times 10^{-3}}{10 \times 10^{-6}} = 10^4 = 40 \text{ dB} \). Since the loss along the cable is 0.5 dB/km, this means that the maximum length of the segment must not exceed \( \frac{40}{0.5} = 80 \text{ km} \). This means that \( m \geq \frac{1200}{80} = 15 \). Thus, the minimum value of \( m \) is 15.

For the rest of the problem, the value of \( m \) found in part (a) is used.

(b) [4 points] If the received signal to noise ratio at the first repeater is 13 dB, what is the noise power at the first repeater?

**Solution:** 13 dB is equivalent to a signal to noise ratio of \( 10^{13/20} \approx 20 \). Thus noise power at the first repeater is given by the signal power at the same repeater divided by 20. With \( m = 15 \), the signal power at the front end of first repeater will be 10 \( \mu \text{W} \), the minimum required for the amplifiers to work. Thus the noise power is \( \frac{10^7}{20} = 0.5 \text{ \( \mu \text{W} \)}. \)

(c) [6 points] If the first repeater is using a binary detector, what is the probability of error \( P_e \) at the first repeater? (Note: This value can be traded for 12 points)

**Solution:** For unipolar NRZ pulses, the probability of error at the first repeater is:

\[
P_{e1} = Q\left(\sqrt{\frac{SNR_1}{2}}\right) = Q\left(\sqrt{20/2}\right) = Q(\sqrt{10}) = Q(3.16) \approx 8 \times 10^{-4}
\]

(d) [6 points] Suppose you use the minimum number of analog signal repeaters by using the value of \( m \) found in part (a). What would be the error probability at the receiver when the cable is used to send binary PAM data using unipolar NRZ scheme.
Solution: If analog signal repeaters are used, the signal to noise ratio at the destination is \( \frac{1}{m} \) times the signal to noise ratio at the first repeater (see analog repeaters in Chapter-9 of text or regenerative repeaters section in Chapter-11 of text). The probability of error is:

\[
P_{e,\text{analog}} = Q\left(\sqrt{\frac{SNR_1}{2m}}\right) = Q\left(\sqrt{\frac{20}{2 \times 15}}\right) = Q\left(\sqrt{\frac{2}{3}}\right) = Q(0.8165) \approx 0.2061
\]

(e) [6 points] If the error probability at the receiver is to be kept at the value computed in part (c) and analog signal repeaters of part (d) are employed, what should be the signal to noise ratio at the first repeater? What should be the new value for \( P_{in} \)?

Solution: To realize the same probability of error, the signal to noise ratio at the first repeater must be increased by a factor \( m \). The new signal to noise ratio needed at the first repeater is, therefore, \( SNR_{1,\text{analog}} = 15 \times 20 = 300 \). Since the noise remains as before, the new signal power at the first repeater is \( 300 \times 0.5 \mu W=150 \mu W=-38.24 \text{ dBW} \). Next, the cable section incurs the same loss as before, i.e., \( .5 \times 80 = 40 \text{ dB} \). Thus, the new transmitted power is \( P_{in,\text{analog}} = -38.24 + 40 = 1.76 \text{ dBW} \).

(f) [6 points] Suppose you use the minimum number of digital signal repeaters (regenerative repeaters) by using the value of \( m \) found in part (a). What would be the error probability in sending binary data send as digital PAM signal with unipolar NRZ scheme.

Problem: The error probability in case of regenerative (digital) repeaters is given by the approximation given in Chapter 11 of the text, and is:

\[
P_{e,\text{digital}} = mQ\left(\sqrt{\frac{SNR_1}{2}}\right) = 15Q\left(\sqrt{\frac{20}{2}}\right) = 15Q(\sqrt{10}) = 15Q(3.16) \approx 0.012
\]

which is much lower than that was obtained in the case of analog repeaters. If we use the bipolar scheme instead, the error probability is \( 15Q(\sqrt{20}) \approx 15 \times 4 \times 10^{-6} = 6 \times 10^{-5} \).

(g) [6 points] If the error probability at the receiver is to be kept at the value computed in part (c) and regenerative signal repeaters of part (f) are employed, what should be the signal to noise ratio at the first repeater? What should be the new value for \( P_{in} \)?

Solution: As in the case of analog repeaters, the signal to noise ratio at the first repeater must be increased but, hopefully, by not as much amount. To compute the new signal to noise ratio needed at the front end of the first repeater, we use, \( 15Q\left(\sqrt{\frac{SNR_{1,\text{digital}}}{2}}\right) = 8 \times 10^{-4} \) which gives \( Q\left(\sqrt{\frac{SNR_{1,\text{digital}}}{2}}\right) = 5.3 \times 10^{-5} \). Using the Q-function table, \( \sqrt{\frac{SNR_{1,\text{digital}}}{2}} \approx 3.9 \) which gives \( SNR_{1,\text{digital}} = 30.42 \). The new signal power at the first repeater is \( 30.42 \times 0.5 \mu W=15.21 \mu W=-48.18 \text{ dBW} \). Finally, the new transmitted power from the source is \( P_{\text{in,digital}} = -48.18 + 40 = -8.18 \text{ dBW} = 0.1521 \text{ W} \). Comparing with the power boost required in the case of analog repeaters, use of digital repeaters mandates significantly less power increase at the transmitter. With bipolar signalling in part (c), the signal-to-noise ratio at the first repeater is 15.21 and the new transmitted power should be .076 W.
Problem 5: Filtering the Phase Modulation [35 points] Consider a single-tone message signal \( x(t) = A_m \cos 2\pi f_m t \) which is used to generate a PM signal \( x_c(t) = A_c \cos[2\pi f_c t + \phi_{\Delta} x(t)] \). We are interested in the envelope and instantaneous frequency deviation of the PM signal when it is passed through a unity-gain ideal bandpass filter. In this problem, use \( \beta = \phi_{\Delta} A_m \) and \( f_m \ll f_C \), as usual. Provide your answers in terms of \( \beta, f_m, A_m, A_c \) and their mathematical functions only.

(a) [7 points] If the PM signal \( x_c(t) \) is passed through a unity gain bandpass filter of bandwidth \( 3f_m \) centered at \( f_c \), write an expression for the envelope of the signal at the output of the bandpass filter.

Solution: The modulated PM signal before the bandpass filtering is given by Equation (18b), page 192 of text:

\[
x_c(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos 2\pi(f_c + nf_m) t
\]

After Bandpass Filtering,

\[
x_{bp}(t) = A_c \sum_{n=-1}^{1} J_n(\beta) \cos 2\pi(f_c + nf_m) t
\]

\[
= A_c J_0(\beta) \cos 2\pi f_c t + A_c J_1(\beta) \cos 2\pi(f_c + f_m) t - A_c J_1(\beta) \cos 2\pi(f_c - f_m) t
\]

\[
= A_c J_0(\beta) \cos 2\pi f_c t - 2A_c J_1(\beta) \sin 2\pi f_m t \sin 2\pi f_c t
\]

From which

\[
x_i(t) = A_c J_0(\beta)
\]

\[
x_q(t) = 2A_c J_1(\beta) \sin 2\pi f_m t
\]

The envelope is,

\[
A(t) = \sqrt{x_i^2(t) + x_q^2(t)}
\]

\[
= A_c \sqrt{J_0^2(\beta) + 4J_1^2(\beta) \sin^2 2\pi f_m t}
\]

(b) [4 points] What is the minimum and maximum value of the envelope in part (a)?

Solution: From the expression for the envelope,

\[
\sin^2 2\pi f_m t = 0 \implies A_{\text{min}} = A_c J_0(\beta)
\]

\[
\sin^2 2\pi f_m t = 1 \implies A_{\text{max}} = A_c \sqrt{J_0^2(\beta) + 4J_1^2(\beta)}
\]
(c) [8 points] If the PM signal \(x_c(t)\) is passed through a unity gain bandpass filter of bandwidth \(2f_m\) centered at \(f_c + 0.5f_m\), write an expression for the envelope of the signal at the output of the bandpass filter.

**Solution:** In this case,

\[
x_{bp}(t) = A_c \sum_{n=0}^{1} J_n(\beta) \cos 2\pi(f_c + nf_m)t
\]

\[
= A_cJ_0(\beta) \cos 2\pi f_c t + A_cJ_1(\beta) \cos 2\pi(f_c + f_m)t
\]

\[
= [A_cJ_0(\beta) + A_cJ_1(\beta) \cos 2\pi f_m t] \cos 2\pi f_c t - A_cJ_1(\beta) \sin 2\pi f_m t \sin 2\pi f_c t
\]

From which

\[
x_i(t) = A_cJ_0(\beta) + A_cJ_1(\beta) \cos 2\pi f_m t
\]

\[
x_q(t) = A_cJ_1(\beta) \sin 2\pi f_m t
\]

Thus,

\[
A(t) = \sqrt{x_i^2(t) + x_q^2(t)}
\]

\[
= A_c \sqrt{J_0^2(\beta) + J_1^2(\beta)} \cos^2 2\pi f_m t + 2J_0(\beta)J_1(\beta) \cos 2\pi f_m t + J_1^2(\beta) \cos^2 2\pi f_m t
\]

\[
= A_c \sqrt{J_0^2(\beta) + J_1^2(\beta) + 2J_0(\beta)J_1(\beta) \cos 2\pi f_m t}
\]

(d) [6 points] What is the minimum and maximum value of the envelope in part (d)?

**Solution:** From the expression for the envelope,

\[
A_{\text{min}} = A_c \sqrt{J_0^2(\beta) + J_1^2(\beta) - 2J_0(\beta)J_1(\beta)} \quad \text{(when } \cos 2\pi f_m t = -1)\]

\[
= A_c |J_0(\beta) - J_1(\beta)|
\]

\[
A_{\text{max}} = A_c \sqrt{J_0^2(\beta) + J_1^2(\beta) + 2J_0(\beta)J_1(\beta)} \quad \text{(when } \cos 2\pi f_m t = +1)\]

\[
= A_c |J_0(\beta) + J_1(\beta)|
\]

(e) [5 points] If the PM signal \(x_c(t)\) is passed through a unity gain bandpass filter of bandwidth \(3f_m\) centered at \(f_c\), write an expression for the instantaneous frequency of the signal at the output of the bandpass filter.

**Solution:** The instantaneous frequency is given by using the in-phase and quadrature components from part (a) as:

\[
f(t) = f_c + \frac{1}{2\pi} \dot{\phi}(t) = f_c + \frac{1}{2\pi} \frac{d}{dt} \arctan \left( \frac{x_q(t)}{x_i(t)} \right)
\]

\[
= f_c + \frac{1}{2\pi} \frac{d}{dt} \tan^{-1} \left( \frac{2A_cJ_1(\beta) \sin 2\pi f_m t}{A_cJ_0(\beta)} \right)
\]
\[ f(t) = f_c + \frac{1}{2\pi} \frac{d}{dt} \tan^{-1} \left( \frac{2J_1(\beta) \sin 2\pi f_m t}{J_0(\beta)} \right) \]
\[ = f_c + \frac{1}{2\pi} \frac{1}{1 + \frac{4J_1^2(\beta) \sin^2 2\pi f_m t}{J_0^2(\beta)}} \frac{d}{dt} \left( \frac{2J_1(\beta) \sin 2\pi f_m t}{J_0(\beta)} \right) \]
\[ = f_c + \frac{1}{2\pi} \frac{J_0^2(\beta)}{A^2(t)} \left( \frac{2J_1(\beta)}{J_0(\beta)} \right)^2 \pi f_m \cos 2\pi f_m t \]
\[ = f_c + 2f_m \cos 2\pi f_m t \frac{J_0(\beta)J_1(\beta)}{A^2(t)} \]

(f) [5 points] If the PM signal \( x_c(t) \) is passed through a unity gain bandpass filter of bandwidth \( 2f_m \) centered at \( f_c + 0.5f_m \), write an expression for the instantaneous frequency of the signal at the output of the bandpass filter.

**Solution:** The instantaneous frequency is given by using the in-phase and quadrature components from part (c) as:

\[ f(t) = f_c + \frac{1}{2\pi} \frac{d}{dt} \arctan \left( \frac{x_q(t)}{x_i(t)} \right) \]
\[ = f_c + \frac{1}{2\pi} \frac{d}{dt} \tan^{-1} \left( \frac{A_cJ_1(\beta) \sin 2\pi f_m t}{A_cJ_0(\beta) + A_cJ_1(\beta) \cos 2\pi f_m t} \right) \]
\[ = f_c + \frac{1}{2\pi} \frac{d}{dt} \tan^{-1} \left( \frac{J_1(\beta) \sin 2\pi f_m t}{J_0(\beta) + J_1(\beta) \cos 2\pi f_m t} \right) \]
A “Standard” normal distribution has zero mean and unity variance.
Numerical values of $Q(k)$ are plotted below for $0 \leq k \leq 7.0$. For larger values of $k$, $Q(k)$ may be approximated by

$$Q(k) \approx \frac{1}{\sqrt{2\pi k}} e^{-k^2/2}$$

which is quite accurate for $k > 3$. 