MATH 230: Probability

Lec # 03

Counting:

Combinatorial Identity:

\( \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \)

Proof: Consider

\[
\binom{n-1}{r-1} + \binom{n-1}{r} = \frac{(n-1)!}{(n-1-r)! (r-1)!} + \frac{(n-1)!}{(n-1-r)! r!}
\]

\[
= \frac{(n-1)!}{(n-r)! (r-1)!} + \frac{(n-1)!}{(n-1-r)! r!}
\]

\[
= \frac{(n-1)!}{(n-r)! r!} \left( \frac{r}{n} + \frac{n-r}{n} \right)
\]

\[
= \binom{n}{r}
\]

Suppose that there \( n \) groups and fix one group at one place. Now there are \( \binom{n-1}{r-1} \) ways to select \( r-1 \) groups from remaining \( n-1 \) groups. Also there are \( \binom{n-1}{r} \) ways to select \( r \) groups that do not contain the group one (fixed one). Therefore there are a total of \( \binom{n}{r} \) ways to select \( r \) groups form \( n \) groups.

Binomial Theorem:

\( (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \)

Combinatorial Proof: Consider the product

\[
\frac{(x + y)(x + y)(x + y) \cdots (x + y)}{n \text{ times}}
\]

Now expand the product using distributive law and group like terms. As for any factor \( (x + y) \), we can choose either \( x \) or \( y \) by multiplying \( (x + y)^n \). Every term can be arranged in form of \( x^k y^{n-k} \), for some \( k = 0,1,2,\ldots,n \). We get the term \( x^k y^{n-k} \) by choosing \( x \) in \( k \) of the \( n \) factors and \( y \) from the remaining \( n-k \) factors. As a result the number of times \( x \) occurs in the product is equal to the number \( \binom{n}{k} \) of the \( k \) subsets of the set of \( n \) factors. Hence
\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

**Example:** Expand \((x + y)^3\)

\[(x + y)^3 = \sum_{i=0}^{3} \binom{3}{i} x^i y^{3-i}\]

\[(x + y)^3 = \binom{3}{0} x^0 y^3 + \binom{3}{1} x^1 y^{2} + \binom{3}{2} x^2 y^1 + \binom{3}{3} x^3 y^0\]

\[(x + y)^3 = y^3 + 3xy^2 + 3x^2y + x^3\]

**Example:** Consider \(n\) digit numbers whose each digit is one of the integers 0, 1, 2, 3, 4, 5, 6, 7, 8 & 9.

(a) How many such numbers are there for which NO two consecutive digits are equal.

First digit can take 10 ways, 2 in 9 ways, and 3rd in 9 ways and so on

\# of digits = 10 \times 9 \times 9 \times 9 \cdots 9 = 9^{n-1}\n
(b) 0 appears as a digit of total \(i\) times \(i = 0, 1, 2, \cdots, n\)

We want to calculate the number of \(n\)-digit numbers where 0 appears \(i\) times. There are \(\binom{n}{i}\) ways to place 0 and remaining \((n - i)\) places have \(9^{n-i}\) choices

\# of ways = \(\binom{n}{i} 9^{n-i}\)

**Multinomial:**

How many number of ways of dividing \(n\) objects into \(r\) groups of sizes \(n_1, n_2, n_3, \cdots, n_r\) respectively and \(n_1 + n_2 + \cdots n_r = n\)?

\# of ways = \(n \binom{n_1}{n_1} \times \binom{n-n_1}{n_2} \times \binom{n-n_1-n_2}{n_3} \times \cdots \times \binom{n-n_1-\cdots (n_r-1)}{n_r}\)

\# of ways = \(\frac{n!}{n_1! \cdots n_r!} = \binom{n}{n_1, n_2, \cdots, n_r}\)

**Example:** If 12 people are to be divided into 3 committees of respective sizes 3, 4, and 5, how many divisions are possible?

\# of ways = \(\binom{12}{3, 4, 5} = \frac{12!}{3!4!5!} = 27720\)
Some Examples:

Example: A student wants to sell 2 books from a collection of 6 Mathematics, 7 Chemistry and 4 Economics books. How many choices are possible if

(a) Both books are of same subjects?

\# of choices = \( \binom{6}{2} + \binom{7}{2} + \binom{4}{2} = 42 \)

(b) Both books are of different subjects?

\# of choices = \( \binom{6}{1} \binom{7}{2} + \binom{6}{1} \binom{4}{1} + \binom{4}{1} \binom{7}{1} = 94 \)

Example (a): In how many ways can 3 boys and 3 girls sit in a row?

\# of ways = \( 6! = 720 \)

(b): In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together?

\# of ways = \( 2 \times 3! \times 3! = 72 \) if boys and girls sit together

(c): In how many ways if only the boys must sit together?

\# of ways = \( 3! \times 4! = 144 \)

(d): In how many ways if no two people of the same sex are allowed to sit together?

\# of ways = \( 2 \times 3! \times 3! = 72 \)

Example: If there are \( N \) people in a room, how many handshakes are possible?

\# of ways = \( \binom{N}{2} = \frac{N!}{2!(N - 2)!} = \frac{1}{2} N(N - 1) \)

Example: In how many ways can \( N \) couple be paired?

(a): There are no restrictions

\# of ways = \( 2^N \binom{N}{2} = \frac{2N!}{2!(2N - 2)!} = N(2N - 1) \)

(b): No same gender pairing

\# of ways choosing couples with same gender pairing

\[ = 2 \times \binom{N}{2} = \frac{2 \times N!}{2!(N - 2)!} = N(N - 1) \]

\# of ways choosing couples without same gender pairing

\[ = N(2N - 1) - N(N - 1) = N^2 \]
(c): No homogeneous pairings in (b) (i.e. No Husband, Wife pairing )

# of ways choosing Homogeneous parings

\[ = N \]

# of ways choosing Non-Homogeneous parings

\[ = N^2 - N \]