Axioms of Probability:

Sample Space:

The set of all possible outcomes of an experiment is known as the *sample space* of the experiment and it is denoted by $S$. If sample space is a set of finite number of elements or unending sequence of the natural number then it is a discrete sample space or else it is continuous sample space. For example

- If the outcome of an experiment to determination of gender of a newborn child then sample space consist on $S = \{boy, girl\}$
- If a fair coin is tossed then sample space will be $S = \{Head, tail\}$
- When a fair dice is rolled, the sample space of this experiment is $S = \{1,2,3,4,5,6\}$
- If the experiment consists of flipping two fair coins then sample space becomes $S = \{HH, HT, TH, TT\}$
- If the experiment is measuring the lifetime of tube lights then sample space is made of all nonnegative real numbers (Continuous Sample Space) $S = \{x| x \in \mathbb{R}^+ \} or \{x|0 \leq x < \infty\}$

Event:

Let $S$ be a Sample space of an experiment. An event is any possible outcome of the experiment. Also it is subset of the sample space. For example

- If the newborn baby is a boy then event is $E = \{boy\}$
- If a fair coin is tossed then a head appears then event becomes $E = \{Head\}$
- A fair dice is rolled, 6 appears then the event is $E = \{6\}$
- If the experiment consists of flipping two fair coins then and event is a Head appears on first coin $E = \{HH, HT\}$
- If $E = \{x|0 \leq x \leq 5\}$ then $E$ is event that tube light does not last longer than 5 years.
Operations on Events:

Union:
Consider $A$ and $B$ be two events of an experiment. $A \cup B$ is an event that will occur if either $A$ or $B$ occurs and it consists of all outcomes that are either in $A$ or in $B$ or in both $A$ and $B$. For example,

- Let $A = \{HT, TH\}$ and $B = \{HH, TH\}$ then $E = A \cup B = \{HH, HT, TH\}$
- Let $A = \{1,2,3,4\}$ and $B = \{3,4,5,\}$ then $E = A \cup B = \{1,2,3,4,5\}$
- Let $A = \{x|0 \leq x \leq 10\}$ and $B = \{x|0 \leq x \leq 25\}$ then $E = A \cup B = \{x|0 \leq x \leq 25\}$

Similarly union of more than two events is defined as

If $A_1, A_2, A_3 \ldots, A_n$ are events of some experiment then

$$E = \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

is an event that consist of all outcomes that are in $A_i$ for at least value of $i = 1,2,3,4,\ldots$

Intersection:
Consider $A$ and $B$ be two events of an experiment. $A \cap B$ is an event that will occur if $A$ and $B$ occurs and it consist of all outcomes that are in both $A$ and $B$. For example,

- Let $A = \{HT, TH\}$ and $B = \{HH, TH\}$ then $E = A \cap B = \{TH\}$
- Let $A = \{1,2,3,4\}$ and $B = \{3,4,5,\}$ then $E = A \cap B = \{3,4\}$
- Let $A = \{x|0 \leq x \leq 10\}$ and $B = \{x|0 \leq x \leq 25\}$ then $E = A \cap B = \{x|0 \leq x \leq 10\}$

Similarly intersection of more than two events is defined as

If $B_1, B_2, B_3 \ldots, B_n$ are events of some experiment then

$$F = \bigcap_{i=1}^{n} B_i = B_1 \cap B_2 \cap \ldots \cap B_n$$

is an event that consist of all outcomes that are all in the events $B_i$, $i = 1,2,3,4,\ldots$

Two events that have no outcome in common are said to be mutually exclusive or disjoint event.

E.g. Let $A = \{1,2,3,4\}$ and $B = \{5,6\}$ then $E = A \cap B = \{\} = \phi$
Complement:
Let A be an event of an experiment. Complement of $A$, denoted by $A^c$ or $A'$, is an event that consist of set of all outcomes in sample space that are not in $A$. For example
- If the experiment consists of flipping two fair coins then sample space becomes $S = \{HH, HT, TH, TT\}$ and suppose that $A = \{HT, TH\}$ then $A^c = \{HH, TT\}$
- A fair dice is rolled, 6 appears then the event is $E = \{6\}$ then $E^c = \{1,2,3,4,5\}$

DeMorgan’s laws:
\[
\left(\bigcup_{i=0}^{n} A_i\right)^c = \bigcap_{i=0}^{n} A_i^c \quad \text{(Prove it!)}
\]
\[
\left(\bigcap_{i=0}^{n} A_i^c\right)^c = \bigcup_{i=0}^{n} A_i^c \quad \text{(Prove it!)}
\]

Identities:
Suppose that $A$, $B$ and $C$ are events of a random experiment then
\[
(A \cup B) \cup C = A \cup (B \cup C)
\]
\[
(A \cap B) \cap C = A \cap (B \cap C)
\]
\[
S^c = \phi
\]
\[
\phi^c = S
\]

Probability:
Let $S$ be the sample space of a random experiment. Let $A$ be an event then probability of $A$ is defined as
\[
P(A) = \frac{\text{Size of } A}{\text{Size of } S}
\]
\[
P(A) = \frac{\# \text{ of success ways}}{\text{Total } \# \text{ of ways}}
\]

Example: If an experiment consists of tossing a fair coin then what is the probability of $E = \{\text{Head}\}$?

- $S = \{\text{Head, Tail}\}$
- $E = \{\text{Head}\}$
- Size of $S = 2$
- Size of $E = 1$
\[ P(E) = \frac{\text{Size of } E}{\text{Size of } S} = \frac{1}{2} \]

**Relative Frequency definition of Probability:**

If an experiment, with sample space \( S \), is repeated \( N \) times under identical conditions and an event \( A \) is occurred \( n \) times then the probability of \( A \) is given as

\[ P(A) = \lim_{N \to \infty} \frac{n}{N} \]

**Example:** Suppose that coin toss experiment is repeated 100 times and head appears 49 times.

\[ P([\text{Head}]) = \frac{49}{100} = 0.49 \]

**Axioms of probability:**

Consider an experiment with sample space \( S \). Assume that for each event \( A \) of the sample space \( P(A) \) is given

Axiom # 1:

\[ 0 \leq P(A) \leq 1 \]

Axiom # 2:

\[ P(S) = 1 \]

Axiom # 3:

For any two (or more than two) mutually exclusive events \( A \) & \( B \)

\[ P(A \cup B) = P(A) + P(B) \]

**Example:** Two coin are tossed what is the probability of

\[ E = \text{Getting exactly one head} \]

\[ F = \text{At least one head} \]

Sol: The sample space is \( S = \{HH, TH, HT, TT\} \) \( |S| = 4 \)

\[ E = \{HT, TH\}, |E| = 2 \]

\[ P(E) = \frac{|E|}{|S|} = \frac{2}{4} = \frac{1}{2} \]

\[ F = \{HT, TH, HH\}, |F| = 3 \]

\[ P(F) = \frac{|E|}{|S|} = \frac{3}{4} \]
Example: A committee of 5 members is to be selected from 6 men and 9 women. Assume each member is chosen at random what is the probability that committee consist of 3 men and 2 women?

\[ P(E) = \frac{^6C_3 \times ^9C_2}{^15C_5} \]