MATH 230 Probability

TUTORIAL PROBLEMS 1

**Problem 1.** How many 3 letter words can be spelled with the 3 letters ABC if repetition is allowed? How many 3 letter words can be spelled with the 3 letters ABC if repetition is not allowed?

**Problem 2.** If 3 letter words are spelled using the 3 letters ABC (with repetition allowed), what fraction of the total possible number of words will have no repetition?

**Problem 3.** How many ways are there to choose 6 symbols from 10 distinct symbols in which the order of choosing matters and repetition is not allowed? Give your answer in factorial notation.

**Answer.** \(10 \times 9 \times 8 \times 7 \times 6 \times 5 = 10!/4!\)

**Problem 4.** How many ways are there to choose 5 symbols from 12 distinct symbols in which the order of choosing matters and repetition is allowed?

**Answer.** \(12^5\)

**Problem 5.** How many ways are there to choose 5 symbols from 12 distinct symbols in which the order of choosing matters and repetition is not allowed?

**Problem 6.** How many ways are there to arrange 13 distinct symbols in a row, no repetition?

**Answer.** \(13!\)

**Problem 7.** You are given thirteen distinct symbols to put into two rows of slots, four slots in the first row and nine in the second. Symbols are not repeated and order matters. How many ways are there to do this?

Answer. \(4! \cdot 9!\)

**Problem 8.** Adil and Bilal go to Yummy’s 36 and order 13 ice cream cones of different flavours. Adil will get four cones and Bilal nine. How many arrangements are possible?

**Answer.** Imagine lining up the 13 ice cream cones in a row and giving the first 4 to Adil and the remaining 9 to Bilal. There are 13! ways of arranging the ice cream cones. We don’t really care about which order the first four are in. Adil still gets the first four ice cream cones, so the 13! has an over-counting by a factor of 4!. Similarly, the order of the remaining 9 cones does not matter but in getting 13! we assumed that it did, so we have also overcounted by a factor of 9! The answer then is

\[
\frac{13!}{4!9!} = \binom{13}{4}
\]

**Problem 9.** How many ways are there to choose 4 symbols from 13 distinct symbols in which the order of choosing does not matter and repetition is not allowed?

**Answer.**

\[
\binom{13}{4} = \frac{13!}{4!9!}
\]

**Problem 10.** Seven juniors and nine seniors wish to be members of a debating team which will have five members. How many different teams are there if each team must contain at least one junior?
**Answer.** The total number of ways to choose five team members from 16 applicants (with no restriction) is

\[ \binom{16}{5} \]

The total number of ways to choose five team members from nine seniors and no juniors is

\[ \binom{9}{5} \]

The total number of ways to choose five team members from 16 applicants, excluding the instance of having no juniors (and thus having at least one junior) is

\[ \binom{16}{5} - \binom{9}{5} \]

**Problem 11.** Seven juniors and nine seniors wish to be members of a debating team which will have five members. How many different teams are there if each team must contain at least one junior and one senior?

**Answer.** There are

\[ \binom{9}{5} \]

ways to choose five seniors and no juniors for the team. There are

\[ \binom{7}{5} \]

ways to choose five juniors and no seniors for the team. There are

\[ \binom{16}{5} \]

ways to choose five team members with no restriction. The expression

\[ \binom{16}{5} - \binom{9}{5} - \binom{7}{5} \]

excludes from the number of committees those that have only juniors or only seniors and thus gives the number with at least one junior and one senior.

**Problem 12.**

a) How many five card hands are there that consist of the ace of hearts, two of hearts, 3 of clubs, 4 of spades, and five of diamonds?

b) How many five card hands are there that contain the ace of hearts, two of hearts, 3 of clubs, and 4 of spades?

c) How many five card hands are there that do not contain the ace of hearts?

**Answer.** a) This describes exactly one unique five card hand.

b) Four cards are fixed, we need to choose one of 48 so 48.

c) There is one way to choose the ace of hearts, there are \( \binom{51}{4} \) ways to choose the remaining four cards.
Problem 13. If we toss four fair coins, and record the result of the toss for each coin, what is the total number of possible results?

Answer. $2 \cdot 2 \cdot 2 \cdot 2$

Problem 14. If we toss four fair coins, and record the result for each individual coin, how many results are there with exactly one tail?

Answer. This is like asking how many arrangements there are of the symbols H,H,H,T. So 4

Problem 15. If we toss four fair coins, and record the result for each individual coin, what fraction of the total number of possible results are there with exactly one tail?

Answer. $\frac{4}{16} = \frac{1}{4}$

Problem 16. If we toss four fair coins, and record the result for each individual coin, how many results are there with exactly two heads and two tails?

Answer. This is like asking how many arrangements there are of the symbols H,H,T,T. If there were four distinct symbols, then the answer would be 4! But if we rearrange the Hs it doesn’t matter and similarly for the Ts. So accounting for overcounting gives $\frac{4!}{(2! \cdot 2!)}$

Problem 17. If we toss four fair coins, and record the result for each individual coin, what fraction of the total number of possible results have exactly two heads and two tails?

Answer. $\frac{4!}{(2! \cdot 2!)} = \frac{4}{2^6} = \frac{1}{2^4} = \frac{1}{16}$

Problem 18. When rolling two fair dice, what fraction of the total number of possible results results in 1 one and 1 two?

Answer. Think of filling one slot using six unique symbols and filling another slot using six unique symbols. So, $6 \cdot 6 = 36$ ways of doing this. The results we are looking for are 1and2 or 2and1, so the answer is $\frac{2}{36}$.

Problem 19. Consider a standard well shuffled 52 card deck of cards. How many three card hands can be dealt from the top of the deck to a player?

Answer. The first card could be one of 52, the second one of 51 and the third one of 50. So there are $52 \cdot 51 \cdot 50$ (ordered) ways of giving the player 3 cards. But it doesn’t matter in what order the cards are held in the player’s hands so any rearrangement of the cards is considered the same hand. So, we have overcounted by 3!. The result is

$$\frac{52 \cdot 51 \cdot 50}{3!} = \frac{52!}{349!} = \binom{52}{3}$$

Problem 20. Consider a standard well shuffled 52 card deck of cards. What fraction of three card hands which can be dealt from the top of the deck to a player contain no aces?

Answer. We have seen that there are $\binom{52}{3}$ possible 3 card hands.

The number of possible 3 card sequences with no aces is $48 \cdot 47 \cdot 46$. This overcounts by 3! since order in a hand does not matter, so number of three card hands with no aces is

$$\frac{48 \cdot 47 \cdot 46}{3!} = \frac{48!}{345!} = \binom{48}{3}$$
The desired answer then is
\[ \frac{4^8}{\binom{52}{5}} \]

**Problem 21.** You are dealt 5 cards from a standard well-shuffled 52 card deck. What fraction of 5 card hands have all cards from the same suit? (This is called a flush).

**Answer.** We can think of constructing a flush as a process of first selecting one suit of four and then from 13 cards of that suit selecting 5. Then there are \( 4 \cdot \binom{13}{5} \) ways of doing this. There are \( \binom{52}{5} \) 5 card hands, so the fraction of 5 card hands which are flushes is
\[ \frac{4 \cdot \binom{13}{5}}{\binom{52}{5}} \]