MATH 230: Probability

Lec # 10

Random Variables:
Definition: A random variable is a function that associates a real number with each element of the sample space. Let \( \Omega \) be sample space then \( X \) is defined as

\[
X : \Omega \to \mathbb{R}
\]

Example: Three fair coins are tossed. Let \( X \) be the random variable that denote the number of heads then \( X \) can take values 0,1,2 and 3 with given probabilities.

\[
P(X = 0) = P(T,T,T) = \frac{1}{8}
\]

\[
P(X = 1) = P((H,T,T),(T,H,T),(T,TH)) = \frac{3}{8}
\]

\[
P(X = 2) = P((H,H,T),(T,H,H),(H,T,H)) = \frac{3}{8}
\]

\[
P(X = 3) = P(H,H,H) = \frac{1}{8}
\]

Example: Independent trials consisting of the flipping of a coin having probability \( p \) of coming up heads are continually performed until either a head occurs or a total of \( n \) flips is made.

Sol: Let \( Y \) be a random variable that counts the number of times coin is tossed then \( Y \) can take values from 1,2,3, \( \cdots \), \( n \) with associated probabilities

\[
P(X = 1) = P(H) = p
\]

\[
P(X = 2) = P(T,H) = p(1 - p)
\]

\[
P(X = 3) = P(T,T,H) = p(1 - p)^2
\]

\[
\vdots
\]

\[
P(X = k) = p(1 - p)^{k-1}
\]

\[
\vdots
\]

\[
P(X = n - 1) = p(1 - p)^{n-2}
\]

\[
P(X = n) = (1 - p)^{n-1}
\]
By law of probability

\[ P\left( \bigcup_{i=1}^{n}(X = i) \right) = \sum_{i=1}^{n} P(X = i) \]

\[ \sum_{i=1}^{n} P(X = i) = \sum_{i=1}^{n-1} p(1-p)^{i-1} + (1-p)^{n-1} \]

\[ \sum_{i=1}^{n} P(X = i) = p\left( \frac{1-(1-p)^{n-1}}{1-(1-p)} \right) + (1-p)^{n-1} = 1 \]

\[ P\left( \bigcup_{i=1}^{n}(X = i) \right) = 1 \]

**Discrete random Variable:**

If the random variable takes the values from set of integers or on at most countable number of possible values than it is said to be discrete random variable.

*Example:* Three balls are randomly chosen from an urn containing 3 white, 3 red, and 5 black balls. Suppose that we win $1 for each white ball selected and lose $1 for each red ball selected.

Find the probabilities of total winnings.

Sol: Let \( X \) be random variable that denotes the total winnings then \( X = \{-3, -2, -1, 0, 1, 2, 3\} \)

\[ P(X = -3) = \frac{\binom{3}{3}}{\binom{11}{3}} = \frac{1}{165} = P(X = 3) \]

\[ P(X = -2) = \frac{\binom{3}{2} \times \binom{5}{1}}{\binom{11}{3}} = \frac{15}{165} = P(X = 2) \]

\[ P(X = -1) = \left( \frac{\binom{3}{1} \times \binom{5}{2}}{\binom{11}{3}} \right) + \left( \frac{\binom{3}{1} \times \binom{3}{2}}{\binom{11}{3}} \right) = \frac{39}{165} = P(X = 1) \]

\[ P(X = 0) = \left( \frac{\binom{3}{1} \times \binom{5}{1} \times \binom{3}{1}}{\binom{11}{3}} \right) + \left( \frac{\binom{5}{3}}{\binom{11}{3}} \right) = \frac{55}{165} \]

**Discrete Probability mass/density function:**

The probability function \( p(a) = P(X = a) \) defined on discrete random variable \( (a \in \mathbb{Z}) \) with the following properties

\[ p(a) = P(X = a) \geq 0 \]
\[
\sum_a p(a) = 1
\]

The function \( p(a) \) is called Probability mass/density function of discrete random variable

**Example:** The probability mass function in the above example can be expressed as

\[
p(x) = \begin{cases} 
1/165, & x = \pm 3 \\
15/165, & x = \pm 2 \\
39/165, & x = \pm 1 \\
55/165, & x = 0 
\end{cases}
\]

**Example:** Three fair coins are tossed. Let \( X \) be the random variable that denote the number of heads. Find probability mass function.

**Sol:**

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x) )</td>
<td>1/8</td>
<td>3/8</td>
<td>3/8</td>
<td>1/8</td>
</tr>
</tbody>
</table>

**Cumulative Probability distribution function:**

The \( F(x) \) of discrete random variable \( X \) with probability mass function \( p(x) \) is given as

\[
F(x) = P(X \leq x) = \sum_x p(X \leq x)
\]

**Example:** Three fair coins are tossed. Let \( X \) be the random variable that denote the number of heads. Find cumulative probability distribution function

\[
F(x) = \begin{cases} 
0, & x < 0 \\
1/8, & 0 \leq x < 1 \\
4/8, & 1 \leq x < 2 \\
7/8, & 2 \leq x < 3 \\
1, & x \geq 3 
\end{cases}
\]
**Expected value:**

Let \( X \) is discrete random variable with probability mass function \( p(x) \). The expected value or mean of \( X \) is given as

\[
\mu = E[X] = \sum_x x \ p(x)
\]

Example: The probability mass function of \( X \) is given by

\[
p(0) = \frac{1}{2} = p(1)
\]

The expected value will be

\[
E[X] = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}
\]

Example: Find \( E[X] \) when a fair die is rolled

Sol: The probability mass function is given as

\[
p(x) = \frac{1}{6}, x = 1,2,3,4,5,6
\]

\[
E[X] = \sum_x x \ p(x) , x = 1,2,3,4,5,6
\]

\[
E[X] = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = \frac{7}{2}
\]

Example: A school class of 120 students is driven in 3 buses to a symphonic performance. There are 36 students in one of the buses, 40 in another, and 44 in the third bus. When the buses arrive, one of the 120 students is randomly chosen. Let \( X \) denote the number of students on the bus of that randomly chosen student, and find \( E[X] \).

Sol: Since students is randomly selected

\[
P(X = 36) = \frac{36}{120}, P(X = 40) = \frac{40}{120}, P(X = 44) = \frac{44}{120}
\]

\[
E[X] = \sum_x x \ p(x) , x = 36,40,44
\]

\[
E[X] = 36 \times \frac{36}{120} + 40 \times \frac{40}{120} + 44 \times \frac{44}{120} = \frac{1208}{30}
\]