MATH 230: Probability

Lec # 11

Random Variables:
Definition: A random variable is a function that associates a real number with each element of the sample space. Let \( \Omega \) be sample space then \( X \) is defined as

\[
X : \ \Omega \rightarrow \mathbb{R}
\]

Discrete random Variable:
If the random variable takes the values from set of integers or on at most countable number of possible values than it is said to be discrete random variable.

Discrete Probability mass/density function:
The probability function \( p(a) = P(X = a) \) defined on discrete random variable \( a \in \mathbb{Z} \) with the following properties

\[
p(a) = P(X = a) \geq 0
\]

\[
\sum_a p(a) = 1
\]

Cumulative Probability distribution function:
The \( F(x) \) of discrete random variable \( X \) with probability mass function \( p(x) \) is given as

\[
F(x) = P(X \leq x) = \sum_x p(X \leq x)
\]

Expected value:
Let \( X \) is discrete random variable with probability mass function \( p(x) \). The expected value or mean of \( X \) is given as

\[
\mu = E[X] = \sum_x x \cdot p(x)
\]

Example: Let \( X \) be random variable that takes values \(-1, 0, 1\) with respective probabilities

\[
P(X = -1) = 0.2, P(X = 0) = 0.5, P(X = 1) = 0.3
\]

Find \( E[X^2] \) and \( E[X] \)
Sol: Let \( Y = X^2 \) \Rightarrow \( E[X^2] = E[Y] \)
\[ E[X] = \sum_x x \, p(x) \quad , x = -1,0,1 \]

\[ E[X] = -1 \times 0.2 + 0 \times 0.5 + 1 \times 0.3 = 0.1 \]

\[ P(Y = 1) = P(X = 1) + P(X = -1) = 0.5 \]

\[ P(Y = 0) = P(X = 0) = 0.5 \]

\[ E[Y] = E[X^2] = \sum_y y \, p(y) \quad , y = 0,1 \]

\[ E[X^2] = 1 \times 0.5 + 0 \times 0.5 = 0.5 \]

Observe that \( E[X^2] \neq (E[X])^2 \)

**Properties of Expectation:**

**Expectation of a function of random variable:**

Let \( X \) be discrete random variable with probability mass function \( p(x_i), i \geq 1 \), then any real valued function \( g(x_i) \) the expectation is given as

\[ E[g(x_i)] = \sum_i g(x_i)p(x_i) \]

**Example:** Let \( X \) be random variable that takes values \(-1, 0, 1\) with respective probabilities

\[ P(X = -1) = 0.2, \quad P(X = 0) = 0.5, \quad P(X = 1) = 0.3 \]

Find \( E[X^2] \)

Sol:

\[ E[X^2] = \sum_x x^2 p(x) = 0.5 \]

**Proposition:**

Let \( X \) be discrete random variable and \( a, b \) are some constants then

\[ E[aX + b] = aE[X] + b \]

**Proof:**
\[ E[aX + b] = \sum_x (ax + b)p(x) \]
\[ E[aX + b] = \sum_x \{axp(x) + bp(x)\} \]
\[ E[aX + b] = a\sum_x xp(x) + b\sum_x p(x) \]
\[ E[aX + b] = aE[X] + b \sum_x xp(x) = E[X], \sum_x p(x) = 1 \]

**Result:** let \( a = 0 \) then
\[ E[b] = b \]

**Variance:**
Let \( X \) be a discrete random variable with probability mass function \( p(x) \) and expected value \( \mu = E[X] \) then variance is defined as
\[ Var[X] = E[(X - E[X])^2] \]

Variance of \( X \) defines that on the average how far the random variable form the average.
\[ Var[X] = E[(X - E[X])^2] \]
\[ Var[X] = \sum_x (x - \mu)^2 p(x) \]
\[ Var[X] = \sum_x (x^2 - 2\mu x + \mu^2)p(x) \]
\[ Var[X] = \sum_x x^2 p(x) - 2\mu \sum_x xp(x) + \mu^2 \sum_x p(x) \]
\[ Var[X] = E[X^2] - 2\mu E[X] + \mu^2 \]
\[ Var[X] = E[X^2] - \mu^2 = E[X^2] - (E[X])^2 \]

Example: Find the \( Var[X] \) where \( X \) represents that a fair die is rolled
Sol: The probability mass function is given as
\[ p(x) = \frac{1}{6}, x = 1,2,3,4,5,6 \]

\[ E[X] = \sum_x x \, p(x) = \frac{7}{2} \]

\[ E[X^2] = \sum_x x^2 \, p(x) = \frac{91}{6} \]

\[ Var[X] = E[X^2] - (E[X])^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12} \]