MATH 230: Probability
Lec # 13

Binomial Random variable:
Consider \( n \) independent trials, each of which is a Bernoulli random variable describing the number of successes is called the Binomial distribution

\[
P(X = i) = \binom{n}{i} p^i q^{n-i}
\]

Also note that

\[
\sum_{i=1}^{n} P(X = i) = \sum_{i=1}^{n} \binom{n}{i} p^i q^{n-i} = (p + q)^n = 1
\]

Expectation of Binomial distribution:

\[
E[X] = \sum_{x} x p(x)
\]

\[
E[X] = \sum_{i=0}^{n} i \times \binom{n}{i} p^i (1-p)^{n-i}
\]

\[
E[X] = \sum_{i=0}^{n} i \times \frac{n!}{i! (n-i)!} \times p^i (1-p)^{n-i}
\]

\[
E[X] = \sum_{i=0}^{n} i \times \frac{n(n-1)!}{i(i-1)! (n-i)!} \times p^i (1-p)^{n-i}
\]

\[
E[X] = n \times \sum_{i=0}^{n} \frac{(n-1)!}{(i-1)! (n-i)!} \times p^i (1-p)^{n-i}
\]

\[
E[X] = n \times p \times \sum_{i=0}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}
\]

\[
E[X] = n \times p
\]

\[
\therefore \sum_{i=0}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i} = 1
\]
Example: Five fair coins are flipped. If the outcomes are assumed independent, find the probability mass function of the number of heads obtained.

Sol: Let $X$ be random variable that denotes the number of heads then $X$ is Binomial random variable with parameters $\left( n = 5 \text{ and } p = \frac{1}{2} \right)$

$$ P(X = i) = \binom{5}{i} \left( \frac{1}{2} \right)^i \left( \frac{1}{2} \right)^{5-i}, i = 0,1,2,3,4,5 $$

Example: The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that at least 10 survive?

Sol: Let $X$ be the number of peoples

$$ P(X \geq 10) = 1 - P(X < 10) $$

$$ P(X \geq 10) = 1 - \sum_{i=0}^{9} \binom{15}{i} (0.4)^i (0.6)^{15-i} = 0.0338 $$

Example: It is known that screws produced by a certain company will be defective with probability 0.01, independently of each other. The company sells the screws in packages of 10 and offers a money-back guarantee that at most 1 of the 10 screws is defective.

What proportion of packages sold must the company replace?

Sol: Let $X$ be the number of the defective items that will be replaced. Then $X$ will be binomial random variable with parameters $n = 10, p = 0.01$ therefore the probability that the package will be replaced is given as

$$ P(2 \leq X \leq 10) = 1 - P(X = 0) - P(X = 1) $$

$$ \Rightarrow 1 - \binom{10}{0} (0.01)^0 (0.99)^{10} - \binom{10}{1} (0.01)^1 (0.99)^9 = 0.04 $$
Computing Binomial Probability:

Consider a Binomial random variable with parameter \( n \) and \( p \). Let \( q = 1 - p \)

\[
\frac{P(X = k + 1)}{P(X = k)} = \frac{{n \choose k + 1} p^{k+1} q^{n-(k+1)}}{{n \choose k} p^k q^{n-k}}
\]

\[
\frac{P(X = k + 1)}{P(X = k)} = \frac{n!}{(k + 1)! (n - (k + 1))!} p^{k+1} q^{n-(k+1)}
\]

\[
\frac{P(X = k + 1)}{P(X = k)} = \frac{p}{q} = \frac{p(n - k)}{q(k + 1)}
\]

\[
\frac{P(X = k + 1)}{P(X = k)} = \frac{p(n - k)}{q(k + 1)}
\]

\[
P(X = k + 1) = \frac{p(n - k)}{q(k + 1)} \times P(X = k)
\]

Example: \( X \sim \text{bin}(6,0.4) \)

\[
P(X = 0) = (0.6)^6 = 0.0467
\]

\[
P(X = 1) = \frac{0.4(6 - 0)}{0.6(0 + 1)} \times 0.0467 = 0.1860
\]

\[
\vdots = \vdots
\]