MATH 230: Probability

Lec # 14

The Poisson Random Variable:

Consider an interval \([a, b]\) in which random events happen with probability \(\lambda\). Now divide the interval into \(n\) parts \((n > \lambda)\). The probability of an event occurred in \(kth\) interval is given as

\[
P(I_k) = \frac{\lambda}{n}, \forall k
\]

Hence \(X = \) total number of the events happen in interval is approximately to binomial distribution with parameter \(p = \frac{\lambda}{n}\) or \(\lambda = np\). The approximation will be better as \(n \to \infty\) so \(X\) will converge to Poisson distribution with parameter \(\lambda\).

Definition: A random variable \(X\) (which take on one value 0,1,2,⋯) is said to be Poisson random variable with parameter \(\lambda > 0\)

\[
p(i) = P(X = i) = \frac{e^{-\lambda} \lambda^i}{i!}, i = 0,1,2,\ldots
\]

The above expression should be equal to one over infinite sum

\[
\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{\infty} \frac{e^{-\lambda} \lambda^i}{i!}
\]

Binomial behaves like Poisson

If random variable \(X\) is binomial random variable with parameter \((n,p)\) when \(n\) is large and \(p\) is small so expected value is of medium size let \(\lambda = np\)

\[
P(X = i) = \binom{n}{i} p^i(1-p)^{n-i}
\]

\[
P(X = i) = \frac{n!}{(n-i)!i!} p^i(1-p)^{n-i}
\]

\[
P(X = i) = \frac{n!}{(n-i)!i!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}
\]

\[
P(X = i) = \frac{n(n-1)(n-2)\cdots(n-i+1) \lambda^i}{n^i} \frac{(1-\frac{\lambda}{n})^n}{i!} \left(1 - \frac{\lambda}{n}\right)^{n-i}
\]
Now $n$ is very large and $p$ is very small hence $\lambda$ will be moderate

\[
\Rightarrow \frac{n(n-1)(n-2)\cdots(n-i+1)}{n^i} = \frac{n(n-1)(n-2)\cdots(n-i+1)}{n \times n \times n \times \cdots \times n} \approx 1 \quad n \to \infty
\]

\[
\Rightarrow \left(1 - \frac{\lambda}{n}\right)^n \approx e^\lambda \quad \text{and} \quad \left(1 - \frac{\lambda}{n}\right)^i \approx 1
\]

\[
P(X = i) = \frac{e^\lambda \lambda^i}{i!}
\]

For $n$ independent trials, each trial occur with probability $p$, are performed when $n$ is large a $p$ is small then the number of success occurring is approximately equal to the Poisson Random variable. Some examples are

*Example*: The number of misprints on a page (or a group of pages) of a book.

*Example*: The number of wrong telephone numbers that are dialed in a day.

*Example*: The number of customers entering a supper store on a given day.

*Example*: Suppose that the probability that an item produced by a certain machine will be defective is 0.1. Find the probability that a sample of 10 items will contain at most 1 defective item.

Sol: let $X$ be the number of defective items $X \sim \text{bin}(10,0.1)$

\[
P(X \leq 1) = \binom{10}{0}(0.1)^0(1-0.1)^{10-0} + \binom{10}{1}(0.1)^1(1-0.1)^{10-1}
\]

\[
P(X \leq 1) = 0.7361
\]

Now Let $Y$ be number of defective items with an average $\lambda = 10 \times 0.1 = 1$. $y \sim \text{Poi}(1)$

\[
P(Y \leq 1) = P(Y = 0) + P(Y = 1)
\]

\[
P(Y \leq 1) = e^{-1} + e^{-1} = 0.7358
\]

*Example*: Suppose that the number of typographical errors on a single page of this book has a Poisson distribution with parameter $\lambda = \frac{1}{2}$. Calculate the probability that there is at least one error on this page.

Sol: let $X$ denote the number of error on this page and $\lambda = \frac{1}{2}$
\[ P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-0.5} = 0.393 \]

**Expectation of Poisson distribution:**

\[
E[X] = \sum_x x \cdot p(x)
\]

\[
E[X] = \sum_{i=0}^{\infty} i \times \frac{e^{-\lambda} \lambda^i}{i!}
\]

\[
E[X] = e^{-\lambda} \sum_{i=0}^{\infty} i \times \frac{\lambda^i}{i!} = \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}
\]

\[
E[X] = \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{(k)!} \quad : \quad k = i - 1
\]

\[
E[X] = \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{(k)!} = \lambda e^{-\lambda} e^\lambda = \lambda
\]

**Variance of Poisson distribution:**

\[
Var(X) = E[X^2] - (E[X])^2
\]

\[
E[X^2] = \sum_x x^2 \cdot p(x)
\]

\[
E[X^2] = \sum_{i=0}^{\infty} i^2 \times \frac{e^{-\lambda} \lambda^i}{i!}
\]

\[
E[X^2] = e^{-\lambda} \sum_{i=0}^{\infty} i^2 \times \frac{\lambda^i}{i!} = \lambda e^{-\lambda} \sum_{i=1}^{\infty} i \times \frac{\lambda^{i-1}}{(i-1)!}
\]

\[
E[X^2] = \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{(k + 1)\lambda^k}{(k)!} \quad : \quad k = i - 1
\]

\[
E[X^2] = \lambda e^{-\lambda} \left\{ \sum_{k=0}^{\infty} \frac{k \lambda^k}{(k)!} + \sum_{k=0}^{\infty} \frac{\lambda^k}{(k)!} \right\}
\]

\[
E[X^2] = \lambda (\lambda + 1)
\]

\[
Var(X) = E[X^2] - (E[X])^2 = \lambda (\lambda + 1) - \lambda^2 = \lambda
\]