MATH 230: Probability
Lec # 17

Continuous Random variable:
Let $X$ be a continuous random variable if there exist and nonnegative function defined on all real values have a property that, for any given set of real numbers $B$ the following statement holds

$$P\{X \in B\} = \int_B f(x) \, dx$$

The function $f$ defined above is called probability mass function or probability mass function for random variable $X$. The $P(X \in B)$ is the area under the curve $f(x)$ and it can be calculated by integrating the probability function $f(x)$ over the set $B$.

The above expression should be equal to one upon integration

$$P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x) \, dx = 1$$

For any given interval $B = [a, b]$ the probability of $X$ is defined as

$$P\{X \in [a, b]\} = P\{a \leq X \leq b\} = \int_{a}^{b} f(x) \, dx$$

The probability of $X$ between $a$ and $b$ shown in the figure.

Now let $a = b$ then probability at that point is zero as the area under function is negligible

$$P\{X = a\} = \int_{a}^{a} f(x) \, dx = F(a) - F(a) = 0$$

Hence the probability of continuous random variable at given fixed point s zero.
Example: Suppose that $X$ is a continuous random variable whose probability density function is given by $f(x)$

(a) Find $C$?
(b) Find $P(X \geq 1)$

\[
f(x) = \begin{cases} 
    C(4x - 2x^2), & 0 < x < 2 \\
    0, & \text{otherwise}
\end{cases}
\]

Sol: Since

\[
P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x) \, dx = 1
\]

\[
P\{X \in (-\infty, \infty)\} = \int_{0}^{2} C(4x - 2x^2) \, dx = 1
\]

\[
C = \frac{3}{8}
\]

Hence Probability mass function is given as

\[
f(x) = \begin{cases} 
    \frac{3}{8}(4x - 2x^2), & 0 < x < 2 \\
    0, & \text{otherwise}
\end{cases}
\]

(b) $P(X \geq 1)$

\[
P\{X \geq 1\} = \int_{1}^{\infty} \frac{3}{8}(4x - 2x^2) \, dx
\]

\[
P\{X \geq 1\} = \int_{1}^{2} \frac{3}{8}(4x - 2x^2) \, dx = \frac{1}{2}
\]

Example: The amount of time in hours that a computer functions before breaking down is a continuous random variable with probability density function given by $f(x)$. What is the probability that

(a) A computer will function between 50 and 150 hours before breaking down?
(b) It will function for fewer than 100 hours?

\[
f(x) = \begin{cases} 
    \lambda e^{-\frac{x}{100}}, & x \geq 0 \\
    0, & x < 0
\end{cases}
\]

Sol: Since total probability is equal to one then

\[
P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} f(x) \, dx = 1
\]

\[
P\{X \in (-\infty, \infty)\} = \int_{-\infty}^{\infty} \lambda e^{-\frac{x}{100}} \, dx = 1
\]
\[ P\{X \in (-\infty, \infty)\} = \int_{0}^{\infty} \lambda e^{-\frac{x}{100}} \, dx = 1 \]

\[ \lambda = \frac{1}{100} \]

(a) The probability of a computer will function between 50 and 150 hours before breaking down is given as

\[ P\{50 \leq X \leq 100\} = \int_{50}^{100} \frac{1}{100} e^{-\frac{x}{100}} \, dx \]

\[ P\{50 \leq X \leq 100\} = e^{-\frac{1}{2}} - e^{-\frac{3}{2}} \approx 0.384 \]

(b) The probability that the computer will function for fewer than 100 hours is calculated as

\[ P\{X < 100\} = \int_{0}^{100} \frac{1}{100} e^{-\frac{x}{100}} \, dx \approx 0.633 \]

**Example:** The lifetime in hours of a certain kind of radio tube is a random variable having a probability density function given by

\[ f(x) = \begin{cases} 
0, & \text{if } x \leq 100 \\
\frac{100}{x^2}, & \text{if } x > 100 
\end{cases} \]

What is the probability that exactly 2 of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation?

Sol: Let \( E_i, i = 1, 2, 3, 4, \ldots \) be the events such that \( i \)th tube is replaced then

\[ P(E_i) = \int_{-\infty}^{150} f(x) \, dx \]

\[ P(E_i) = \int_{100}^{150} \frac{100}{x^2} \, dx = \frac{1}{3} \]

We assumed that all tubes open independently

\[ P(2 \text{ of 5}) = \binom{5}{2} \times \left(\frac{1}{3}\right)^2 \times \left(\frac{2}{3}\right)^3 = \frac{80}{243} = 0.34 \]

**Example:** Find the value of \( c \) in given probability mass function \( f(x) \)

\[ f(x) = \begin{cases} 
0, & \text{if } x \leq 0 \\
cxe^{-x}, & \text{if } x > 0 
\end{cases} \]

Sol: the total probability should be equal to 1
\[ P(X) = \int_{-\infty}^{\infty} f(x)dx = 1 \]

\[ P(X) = \int_{0}^{\infty} cxe^{-x}dx = 1 \Rightarrow c = 1 \]

Hence

\[
f(x) = \begin{cases} 
0, & x \leq 0 \\
x e^{-x}, & x > 0
\end{cases}
\]