MATH 230: Probability

Lec # 18

Expectation & Variance:

Expectation:

Let $X$ is continuous random variable with probability mass function $f(x)$. The expected value or mean of $X$ is given as

$$
\mu = E[X] = \int_{-\infty}^{\infty} x f(x)dx
$$

**Example:** Find $E[X]$ when the density function of $X$ is

$$f(x) = \begin{cases} 
2x, & 0 < x < 1 \\
0, & \text{otherwise}
\end{cases}$$

Sol:

$$E[X] = \int_{-\infty}^{\infty} x f(x)dx$$

$$E[X] = \int_{-\infty}^{\infty} x \times 2x \, dx = \int_{0}^{1} 2x^2 \, dx = \frac{2}{3}$$

**Example:** Find the expected value of continuous random variable with density function

$$f(x) = \begin{cases} 
|x| \times 10, & -2 \leq x \leq 4 \\
0, & \text{otherwise}
\end{cases}$$

Sol:

$$E[X] = \int_{-\infty}^{\infty} x f(x)dx$$

$$E[X] = \int_{-\infty}^{\infty} x \times \frac{|x|}{10} \, dx = \int_{-2}^{0} -\frac{x^2}{10} \, dx + \int_{0}^{4} \frac{x^2}{10} \, dx = \frac{28}{15}$$

Variance:

Let $X$ be a continuous random variable with probability mass function $f(x)$ and expected value $\mu = E[X]$ then variance is defined as

$$Var[X] = E[(X - E[X])^2]$$

$$Var[X] = E[X^2] - (E[X])^2$$

Variance of $X$ defines that on the average how far the random variable form the average.
Example: Find $\text{Var}[X]$ when the density function of $X$ is

$$f(x) = \begin{cases} 
2x, & 0 < x < 1 \\
0, & \text{otherwise} 
\end{cases}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) \, dx$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \times 2x \, dx = \int_{0}^{1} 2x^3 \, dx = \frac{1}{2}$$

$$\text{Var}[X] = E[X^2] - (E[X])^2$$

$$\text{Var}[X] = \left(\frac{1}{2}\right) - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

Expectation of a function of $X$

Let $X$ be continuous random variable with probability mass function $f(x)$, then any real valued function $g(x)$ the expectation is given as

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x) \, dx$$

Example: Let $X$ be random variable with probability density function

$$f(x) = \begin{cases} 
1, & 0 < x < 1 \\
0, & \text{otherwise} 
\end{cases}$$

Find $E[e^X]$ 

Sol: Let $g(x) = e^x$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x) \, dx$$

$$E[e^X] = \int_{-\infty}^{\infty} e^x \times 1 \, dx$$

$$E[e^X] = \int_{0}^{1} e^x \, dx = e - 1$$

Uniform random variable:

Let $X$ be a random variable then $X$ is said to be uniform random variable if the probability density function is given as

$$f(x) = \begin{cases} 
1, & 0 < x < 1 \\
0, & \text{otherwise} 
\end{cases}$$

It is denoted by $X \sim U(0,1)$
In general for a given interval \((\alpha, \beta)\). \(X \sim U(\alpha, \beta)\) is defined as

\[
f(x) = \begin{cases} 
\frac{1}{\beta - \alpha}, & \alpha < x < \beta \\
0, & \text{otherwise}
\end{cases}
\]

And Cumulative function \(F(a) = \int_{-\infty}^{a} f(x)\) on interval \((\alpha, \beta)\) is given as

\[
F(a) = \begin{cases} 
0, & a \leq \alpha \\
\frac{a - \alpha}{\beta - \alpha}, & \alpha < a < \beta \\
1, & a \geq \beta
\end{cases}
\]

Example: Let \(X \sim U(0, 10)\) Find \(P(X < 3)\) and \(P(3 < X < 8)\)

\[
f(x) = \begin{cases} 
\frac{1}{10}, & 0 < x < 10 \\
0, & \text{otherwise}
\end{cases}
\]

\[
P(X < 3) = \int_{-\infty}^{3} f(x) = \int_{0}^{3} \frac{1}{10} \, dx = \frac{3}{10}
\]
\[ P(3 < X < 8) = \int_{3}^{8} f(x) = \int_{3}^{8} \frac{1}{10} \, dx = \frac{1}{2} \]

**Example:** Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits
(a) Less than 5 minutes for a bus;
(b) More than 10 minutes for a bus

Sol: let \( X \) be the number of minutes past 7 that passenger comes at bus stop. \( X \) is uniformly distributed in \((0,30)\). Passenger has to wait less than 5 minutes if and only if he comes between 7:10 and 7:15 or between 7:25 and 7:3. Hence

\[ P(10 < X < 15) + P(25 < X < 30) = \int_{10}^{15} \frac{1}{30} \, dx + \int_{25}^{30} \frac{1}{30} \, dx \]

Probability = \( \frac{1}{3} \)

Similarly he would have to wait more than 10 minutes when he will come between 7:00 and 7:05 or between 7:15 and 7:20

\[ P(0 < X < 05) + P(15 < X < 20) = \int_{0}^{5} \frac{1}{30} \, dx + \int_{15}^{20} \frac{1}{30} \, dx \]

Probability = \( \frac{1}{3} \)

**Expectation of Uniform random variable:**

Let \( X \sim U(\alpha, \beta) \)

\[ E[X] = \int_{-\infty}^{\infty} x \, f(x) \, dx \]

\[ E[X] = \int_{-\infty}^{\infty} x \frac{1}{\beta - \alpha} \, dx = \int_{\alpha}^{\beta} x \frac{1}{\beta - \alpha} \, dx \]

\[ E[X] = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} x \, dx = \frac{\beta + \alpha}{2} \]

**Variance of Uniform distribution:**

\[ Var(X) = E[X^2] - (E[X])^2 \]

\[ E[X^2] = \int_{-\infty}^{\infty} x^2 \, f(x) \, dx \]
\[ E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\beta - \alpha} \, dx = \int_{\alpha}^{\beta} x^2 \frac{1}{\beta - \alpha} \, dx \]

\[ E[X^2] = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} x^2 \, dx = \frac{\beta^2 + \alpha \beta + \alpha^2}{3} \]

\[ Var(X) = E[X^2] - (E[X])^2 = \frac{\beta^2 + \alpha \beta + \alpha^2}{3} - \left( \frac{\beta + \alpha}{2} \right)^2 \]

\[ Var(X) = \frac{(\beta - \alpha)^2}{12} \]