MATH 230: Probability

Lec # 23

Memoryless Property:
Let $X$ be nonnegative ransom variable, $X$ is said to be memoryless if for any $s, t \geq 0$ following equation holds

$$P\{X > s + t | X > t\} = P\{X > s\}, \quad \forall s, t \geq 0$$

$$\frac{P\{X > s + t, X > t\}}{P\{X > t\}} = P\{X > s\}$$

$$P\{X > s + t, X > t\} = P\{X > t\}P\{X > s\}$$

If $X$ is the life time of the some instrument then the above equation explains that the probability of the instrument survives for at least $s + t$ years given that it has survived for $t$ years is same as the probability for the survives for at least $s$ years.

Example: Prove that exponential distribution is memoryless.

Sol:

$$P\{X > s + t, X > t\} = P\{X > t\}P\{X > s\}$$

$$P\{X > s + t, X > t\} = e^{-\lambda(s+t)}, \quad \forall s, t \geq 0$$

$$P\{X > s + t, X > t\} = e^{-\lambda s} \times e^{-\lambda t}, \quad \forall s, t \geq 0$$

$$P\{X > s + t, X > t\} = P\{X > t\}P\{X > s\}$$

Let

$$F(x) = P\{X \leq x\} = 1 - e^{-\lambda x}$$

$$\bar{F}(x) = P\{X > x\} = 1 - F(x) = e^{-\lambda x}$$

$$\bar{F}(s + t) = \bar{F}(s) \times \bar{F}(t)$$
Example: Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5000-mile trip, what is the probability that he or she will be able to complete the trip without having to replace the car battery?

Sol: Let $X$ be the life time of battery (in miles) then $\lambda = \frac{1}{10}$. Since $X$ is exponentially distributed the it follows from memoryless property of the exponential distribution

$$P\{X > 5\} = 1 - F(5) = e^{-5\lambda} = e^{-0.5} \approx 0.604$$

Now consider $X$ is not exponentially distributed then

$$P\{X > t + 5|X > t\} = \frac{1 - F(t + 5)}{1 - F(t)}$$

Where $t$ is number of the miles that battery had been used before start of the trip

**Hazard Rate Functions:**

Suppose a positive continuous random variable $X$ that represents the lifetime of some item. Let $X$ have distribution $F$ and density function $f$. The hazard rate function $\lambda(t)$ of the distribution $F$ is defined as

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)} \quad \text{where} \quad \bar{F}(t) = 1 - F(t)$$

Let an item has survived for a time $t$ years. We need the probability that it will not survives for additional time $dt$.

$$P\{X \in (t, t + dt)|X > t\} = \frac{P\{X \in (t, t + dt), X > t\}}{P\{X > t\}}$$

$$P\{X \in (t, t + dt)|X > t\} = \frac{P\{X \in (t, t + dt)\}}{P\{X > t\}}$$

$$P\{X \in (t, t + dt)|X > t\} \approx \frac{f(t)}{\bar{F}(t)} dt$$

Hence $\lambda(t)$ is the conditional probability intensity that a $t$ year old item will fail.

Example: Show that failure rate or $\lambda(t)$ of an item following the exponential distribution is constant.

Sol: $X \sim \text{Exp}(\lambda)$

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)} \quad \text{where} \quad \bar{F}(t) = 1 - F(t)$$
\[ \lambda(t) = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda \]

Thus the failure rate of exponential distribution is same as the parameter \( \lambda \) of the distribution.

**Finding distribution using hazard rate function:**

We can use the \( \lambda(t) \) to fine the distribution \( F \).

\[
\lambda(t) = \frac{f(t)}{F(t)} \quad \text{where} \quad F(t) = 1 - F(t)
\]

\[
\lambda(t) = \frac{\frac{d}{dt}F(t)}{1 - F(t)} \quad \because f(t) = \frac{d}{dt}F(t)
\]

Integrating on both sides

\[
\log(1 - F(t)) = -\int_0^t \lambda(t) dt
\]

\[
F(t) = 1 - \exp \left\{-\int_0^t \lambda(t) dt \right\}
\]

\[
f(t) = \frac{d}{dt}F(t)
\]

**Example:** Let a random variable \( X \) has linear hazard rate function \( \lambda(t) = a + bt \). Find the distribution of \( X \)?

Sol:

\[
\lambda(t) = a + bt
\]

\[
F(t) = 1 - \exp \left\{-\int_0^t (a + bt) dt \right\}
\]

\[
F(t) = 1 - \exp \left\{-\int_0^t (a + bt) dt \right\}
\]

\[
f(t) = \frac{d}{dt}\left\{1 - \exp \left\{-\int_0^t (a + bt) dt \right\}\right\}
\]

\[
f(t) = (a + bt)\left(\exp \left\{-\left(at + \frac{bt^2}{2}\right)\right\}\right), \quad t > 0
\]
Example: Let $\lambda(t)$ be the rate of failure of an item. Find the probability an A-year-old item will survive until age B?

Sol:

\[ P\{A \text{- year \ - old reaches age } B \} = P\{\text{life time } B \mid \text{life time } > A\} \]

\[ P = \frac{1 - F(B)}{1 - F(A)} \]

\[ P = \frac{\exp\left\{- \int_{0}^{B} \lambda(t)\,dt\right\}}{\exp\left\{- \int_{0}^{A} \lambda(t)\,dt\right\}} \]

\[ P = \exp\left\{- \int_{A}^{B} \lambda(t)\,dt\right\} \]

Example: Suppose that the life distribution of an item has the hazard rate function $\lambda(t) = t^3, t > 0$. What is the probability that it survives

(a) to age 2

(b) the item’s lifetime is between 0.4 and 1.4

Sol: (a)

\[ \lambda(t) = t^3, t > 0 \]

\[ F(t) = 1 - \exp\left\{- \int_{0}^{t} \lambda(t)\,dt\right\} \]

\[ F(t) = 1 - \exp\left\{- \int_{0}^{t} t^3\,dt\right\} \]

\[ P\{X \geq 2\} = 1 - F(2) \]

\[ F(t) = 1 - \exp\left\{- \int_{0}^{2} t^3\,dt\right\} \]

\[ P\{X \geq 2\} = \exp\left\{- \int_{0}^{2} t^3\,dt\right\} = e^{-4} = 0.0183 \]

(b) the item will survive from $t = 0.4$ to $t = 1.4$

\[ P(0.4 < X < 1.4) = \exp\left\{- \int_{0.4}^{1.4} t^3\,dt\right\} \]

\[ = \exp\left\{- \int_{0.4}^{1.4} t^3\,dt\right\} = 0.3851 \]