MATH 230: Probability  
Lec # 26

Independent random variable:
Let $X$ and $Y$ be two random variables. Random variables are said to be independent if for any two sets of real numbers $A$ and $B$

$$P\{X \in A, Y \in B\} = P\{X \in A\}P\{Y \in B\}$$

$$P\{X \leq a, Y \leq b\} = P\{X \leq a\}P\{Y \leq b\}$$

Let $X$ and $Y$ be random variables with joint density function $f(x, y)$ then $X$ and $Y$ are independent if the following condition holds

$$f(x, y) = f_X(x)f_Y(y) \; \forall \; x, y$$

Example: Suppose that $n + m$ independent trials having a common probability of success $p$ are performed. If $X$ is the number of successes in the first $n$ trials, and $Y$ is the number of successes in the final $m$ trials, then show that $X$ and $Y$ are independent.

Sol: Since knowing the number of successes in the first $n$ trials does not affect the distribution of the number of successes in the final $m$ trials. Hence $X$ and $Y$ are independent

$$P\{X = x, Y = y\} = \binom{n}{x} p^x (1-p)^{n-x} \times \binom{m}{y} p^y (1-p)^{m-y} \; \forall \; 0 \leq x \leq n \; 0 \leq y \leq m$$

Example: Suppose that the number of people who enter a post office on a given day is a Poisson random variable with parameter $\lambda$. Show that if each person who enters the post office is a male with probability $p$ and a female with probability $1 - p$, then the number of males and females entering the post office are independent Poisson random variables with respective parameters $\lambda p$ and $\lambda (1 - p)$.

Sol: let $X$ and $Y$ be random variables that represents the number of males and females entering in the post office respectively. We will use the conditional probability to prove independence

$$P(E) = P(E|A)P(A) + P(E|A^c)P(A^c)$$

$$P\{X = i, Y = j\} = P\{X = i, Y = j|X + Y = i + j\}P\{X + Y = i + j\} + P\{X = i, Y = j|X + Y \neq i + j\}P\{X + Y \neq i + j\}$$
Since the probability of number of males and number females given that number people is not equal to the sum of males and females is equal to zero

\[ P\{X = i, Y = j | X + Y \neq i + j\} = 0 \]

\[ P\{X = i, Y = j\} = P\{X = i, Y = j | X + Y = i + j\} P\{X + Y = i + j\} \]

The probability of the total number of the people is given as

\[ P\{X + Y = i + j\} = e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!} \]

Given that \( i + j \) people enter the post office, since each person entering will be male (or female) with probability \( p \) it follows that probability that exact \( i \) of them male (or \( j \) of them female) is just the binomial probability

\[ P\{X = i, Y = j | X + Y = i + j\} = \binom{i+j}{i} p^i (1-p)^j \]

Therefore

\[ P\{X = i, Y = j\} = P\{X = i, Y = j | X + Y = i + j\} P\{X + Y = i + j\} \]

\[ P\{X = i, Y = j\} = \binom{i+j}{i} p^i (1-p)^j \times e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!} \]

\[ P\{X = i, Y = j\} = \frac{e^{-\lambda p} (\lambda p)^i}{i!} e^{-\lambda (1-p)} \left\{ \frac{[\lambda(1-p)]^j}{j!} \right\} \]

Now

\[ P\{X = i\} = \frac{e^{-\lambda p} (\lambda p)^i}{i!} \sum_j e^{-\lambda (1-p)} \left\{ \frac{[\lambda(1-p)]^j}{j!} \right\} \]

\[ P\{X = i\} = \frac{e^{-\lambda p} (\lambda p)^i}{i!} \sum_j e^{-\lambda (1-p)} \left\{ \frac{[\lambda(1-p)]^j}{j!} \right\} \]

\[ P\{Y = j\} = \sum_i \frac{e^{-\lambda p} (\lambda p)^i}{i!} e^{-\lambda (1-p)} \left\{ \frac{[\lambda(1-p)]^j}{j!} \right\} \]

\[ P\{Y = j\} = e^{-\lambda (1-p)} \left\{ \frac{[\lambda(1-p)]^j}{j!} \right\} \]

\[ P\{X = i, Y = j\} = P\{X = i\} P\{Y = j\} \]