Problem 1. You select seven cards from a standard 52 card deck. Let $X$ be the number of aces in the hand selected. What is the pmf for $X$?

Answer. All seven card hands are equally likely. There are $\binom{52}{7}$ such hands.

There are four aces, so the number of ways to choose $x$ aces and $7-x$ other cards from 48 is

$\binom{4}{x}\binom{48}{7-x}$

As all hands are equally likely, we divide by the total number of seven card hands to get the probability of getting $x$ aces in the hand

$$\frac{\binom{4}{x}\binom{48}{7-x}}{\binom{52}{7}}$$

Then the pmf for $X$ is

$$f_X(x) = \begin{cases} \frac{\binom{4}{x}\binom{48}{7-x}}{\binom{52}{7}}, & x = 0, 1, 2, 3, 4 \\ 0, & \text{else} \end{cases}$$

Problem 2. You select seven cards from a standard 52 card deck. Let $Y$ be the number of queens in the hand selected. What is the pmf for $Y$?

Answer. Same reasoning as before.

$$f_Y(y) = \begin{cases} \frac{\binom{4}{y}\binom{48}{7-y}}{\binom{52}{7}}, & y = 0, 1, 2, 3, 4 \\ 0, & \text{else} \end{cases}$$

Problem 3. Find the joint pmf $f_{XY}$ for $X$ and $Y$.

Answer. There are $\binom{52}{7}$ 7-card hands. All such hands are equally likely.

The number of ways to get $x$ aces from four aces, $y$ queens from four queens and then $7-x-y$ other cards from 44 cards that are not aces or queens is $\binom{4}{x}\binom{4}{y}\binom{44}{7-x-y}$.

The joint pmf then is

$$f_{XY}(x, y) = \begin{cases} \frac{\binom{4}{x}\binom{4}{y}\binom{44}{7-x-y}}{\binom{52}{7}}, & \text{for } x, y = 0, 1, 2, 3, 4 \text{ and } x+y \leq 7 \\ 0, & \text{else} \end{cases}$$

Problem 4. Are $X$ and $Y$ statistically independent?
Problem 5. Suppose that there are three sections of probability. Each of the sections has sophomores and juniors in it. These are distributed according to the table

<table>
<thead>
<tr>
<th>Section 1</th>
<th>Section 2</th>
<th>Section 3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sophomores</td>
<td>20</td>
<td>40</td>
<td>60</td>
</tr>
<tr>
<td>Juniors</td>
<td>50</td>
<td>30</td>
<td>20</td>
</tr>
<tr>
<td>Total</td>
<td>70</td>
<td>70</td>
<td>80</td>
</tr>
</tbody>
</table>

Let $X$ be a random variable that takes the value 1, 2, 3 depending on what section a randomly chosen sophomore taking probability is in.

Let $Y$ be a random variable that takes the value 1, 2, 3 depending on what section a randomly chosen junior taking probability is in.

Find $f_X$, $f_Y$, and $f_{XY}$.

Problem 6. Suppose that rvs $X$ and $Y$ have a joint distribution given by

$$f_{XY}(x, y) = \begin{cases} 
  2x + 2y & \text{for } 0 \leq x \leq y \leq 1 \\
  0 & \text{else}
\end{cases}$$

Find $f_X, f_Y$. Are $X$ and $Y$ independent?

Answer. $X$ and $Y$ are dependent. After finding $f_X$ and $f_Y$ we see that $f_{XY} \neq f_X \cdot f_Y$.

Problem 7. Suppose that $X$ and $Y$ have a joint cdf $F_{XY}$ defined by

$$F_{XY}(x, y) = \begin{cases} 
  (1 - e^{-x})(1 - e^{-y}) & \text{for } x, y > 0 \\
  0 & \text{else}
\end{cases}$$

What are the marginal pdfs $f_X$ and $f_Y$?

Answer. If we differentiate $F_{XY}$ wrt $x$ and $y$ we get the joint pdf

$$f_{XY}(x, y) = \begin{cases} 
  e^{-x} \cdot e^{-y} & \text{for } x, y > 0 \\
  0 & \text{else}
\end{cases}$$

We can now find the individual pdfs by integrating out the other variable.

$$f_X(x) = \begin{cases} 
  e^{-x} & \text{for } x > 0 \\
  0 & \text{else}
\end{cases}$$

and

$$f_Y(y) = \begin{cases} 
  e^{-y} & \text{for } y > 0 \\
  0 & \text{else}
\end{cases}$$

Problem 8. Suppose that $X$ and $Y$ have a joint cdf $F_{XY}$ defined by

$$F_{XY}(x, y) = \begin{cases} 
  (1 - e^{-x})(1 - e^{-y}) & \text{for } x, y > 0 \\
  0 & \text{else}
\end{cases}$$

What are the marginal cdfs $F_X$ and $F_Y$?

Answer. We find the marginal distributions from the joint cdf by letting the other variable go to infinity.
\[ F_X(x, y) = \begin{cases} 1 - e^{-x} & \text{for } x > 0 \\ 0 & \text{else} \end{cases} \]

\[ F_Y(x, y) = \begin{cases} 1 - e^{-y} & \text{for } y > 0 \\ 0 & \text{else} \end{cases} \]

**Problem 9.** Are \( X \) and \( Y \) in the previous problem statistically independent?

**Answer.** Yes, \( F_{XY} = F_X \cdot F_Y \)

**Problem 10.** Suppose that \( X \) and \( Y \) have a joint distribution \( f_{XY} \) defined by

\[ f_{XY} = \begin{cases} \frac{1}{x!y!} \lambda^x \mu^y e^{-(\lambda+\mu)} & \text{for } x, y = 0, 1, 2, 3, \ldots \\ 0 & \text{else} \end{cases} \]

Are \( X \) and \( Y \) independent?