EXAMPLES 8

Note 1. Standard normal

\[ f_X(x) = \frac{1}{\sqrt{2\pi}e^{-\frac{1}{2}x^2}} \]

\[ E[X] = \mu_X = 0, \ Var[X] = \sigma_X^2 = 1, \ \sigma_X = 1 \]

Problem 1. Suppose that \( Z \sim N(0, 1) \). \( P(Z \geq 0) \) is what?

Answer. Distribution is symmetric about zero, so 50%.

Problem 2. \( Z \) values are given in tables which show the cumulative distribution \( F_Z(x) = P(Z \leq x) \). What is \( P(Z \leq 0.1) \)?

Answer. From the Z-table: 0.5398

Problem 3. What is \( P(-0.3 \leq Z \leq 0.1) \)?

Answer. From the Z-table: 0.5398 - 0.0013 = 0.5385

Problem 4. Suppose that \( X \) is a normal random variable with expectation 1 and variance 4.

What is \( P(0 \leq X \leq 5) \)?

Answer. The tables only work for the standard normal with expectation zero and std dev 1. So use the new random variable

\[ Z = \frac{X - \mu}{\sigma} \]

Then \( X = 0 \Rightarrow Z = -1/2 \) and \( X = 5 \Rightarrow Z = (5 - 1)/2 = 2. \)

Then from the tables \( P(0 \leq X \leq 5) = P(-1/2 \leq Z \leq 2) = 0.9772 - 0.3085 = 0.6687 \)

Problem 5. The amount of ketchup that bottles at Jammin’ Java dispense in one squirt is approximately normally distributed with mean 9 grams and std dev 1 gram. Out of 500 squirts, approximately how many will be of 10 grams or more?

Answer. The probability of a squirt being less than 1 std dev greater than the expectation is \( P(Z \leq 1) = 0.8413 \) from the table. Then the probability of being greater is 1 - 0.8413 = 0.1587

Then the number of squirts of more than 10 grams will be about \( 500 \times 0.1587 = 79.35 \)

Note 2. Sometimes we can use the normal to approximate ranges of the binomial.

Problem 6. What is the probability of getting 57 \( H \)s when flipping a fair coin 100 times?

Answer.

\[ \binom{100}{57}(1/2)^{57}(1/2)^{43} \]

Problem 7. What is the probability of at most 57 \( H \)s in 100 flips?

Answer.

\[ \sum_{n=1}^{57} \binom{100}{k}(1/2)^k(1/2)^{100-k} \]
Note 3. For a binomial distribution, recall $\mu = np$, $\sigma^2 = np(1 - p)$.

Note 4. Approximating the binomial by a normal. Rule of thumb: If $np \geq 5$ and $n(1 - p) \geq 5$ then the normal curve can be used to approximate the binomial.

Problem 8. Approximate the probability of at most 57 flips with the normal.

Answer. $n = 100$, $p = 0.5$, $1 - p = 0.5$ so $np = 50 \geq 5$ and $n(1 - p) = 50 \geq 5$. So, we can approximate using the normal distribution.

The normal distribution we want to use is the one with $\mu = np = 50$ and $\sigma^2 = np(1 - p) = 25$ or $\sigma = 5$. We want $P(X \leq 57)$.

Converting to the standard normal $Z$ we want $P\left(Z \leq \frac{57 - 50}{5}\right)$.

Actually, no. We are approximating a histogram with a continuous curve so there will be a continuity correction of plus or minus 0.5 to include the entire rectangle in the calculations.

So, we actually want $P\left(Z \leq \frac{57.5 - 50}{5}\right) = P(Z \leq 1.5) = 0.9932$

Note 5. A note on the continuity error correction.

The binomial is a discrete distribution and then we are trying to approximate by a normal distribution with the same parameters. The correction is to either add or subtract 0.5 from each discrete value so as to fill in the gaps.

Problem 9. Approximate the probability of getting exactly 57 Hs using the normal distribution.

Answer. The actual binomial $X$ has $\mu = np = 50$ and $\sigma^2 = 25 \implies \sigma = 5$ as we have seen before. We have also checked the rule of thumb to see that a normal approximation is reasonable.

The probability $P(X = 57) = P(56.5 < X < 57.5)$ since $X$ can only take discrete nonnegative integer values. Then we approximate by a normal distribution using the range 56.5 to 57.5

Converting to a standard normal the range is 1.1 to 1.5 giving

$P(X = 57) \approx F_Z(1.5) - F_Z(1.1) = 0.9332 - 0.8643 = 0.0689$

Problem 10. Approximate the probability of getting more 57 Hs using the normal distribution.

Answer. The actual binomial random variable can take values 56, 57, 58, ... so $P(X > 57)$ and $P(X > 57.5)$ are the same. 57.5 in terms of the standard normal is 1.5 so

$P(X > 57) \approx 1 - F_Z(1.5) = 1 - 0.9332 = 0.0668$

Problem 11. Approximate the probability of getting 57 or more Hs using the normal distribution.

Answer. $P(X \geq 57) = P(X \geq 56.5)$ so when using the normal we will use the range 56.5 to infinity. In terms of the standard normal, 56.5 corresponds to 1.1 so

$P(X \geq 57) \approx 1 - F_Z(1.1) = 1 - 0.8643 = 0.1357$

Problem 12. Approximate the probability of less than 57 Hs using the normal distribution.

Answer. For the original binomial random variable $X$, $P(X < 57) = P(X \leq 56.5)$. So when approximating with a normal random variable we will use the range minus infinity to 56.5 In terms of the standard normal variable, 56.5 corresponds to 1.1 so
\[ P(X < 57) \approx F_Z(1.1) = 0.8643 \]

**Problem 13.** Approximate the probability of at most 57 Hs using the normal distribution.

**Answer.** For the original binomial random variable \( X \), \( P(X \leq 57) = P(X < 57.5) \). So when approximating with a normal random variable we will use the range minus infinity to 57.5 In terms of the standard normal variable, 57.5 corresponds to 1.5 so

\[ P(X < 57) \approx F_Z(1.5) = 0.9932 \]