DO NOT OPEN THIS EXAM UNTIL TOLD TO DO SO.

The instructions below must be followed strictly. Failure to do so can result in serious grade loss.

⇒ You may not
• talk to anyone once the exam begins.
• leave the examination room and then return.

⇒ Keep your eyes on your own paper.

⇒ Read all questions very carefully before answering them.

Instructions

1. It is an OPEN BOOK and OPEN NOTES examination but you are not allowed to share any thing.
2. You are requested to write within the boxes provided. Any solution outside the box will not be marked.
3. You should use a simple pencil so that you can erase at will otherwise you will spoil the answer box.
4. Count the number of pages before you start, they should be 11 other than this cover sheet. The total number of questions is 15. Do not remove any pages from this book. Write your roll number on every page NOW.
5. Any space outside the boxes may be used for rough work.
6. Use of a Calculator or a personal computer is allowed.
Question 1:  
marks: 4

Write an expression, using only the symbols $x$, $($, $)$, $<$, $>$, $\lor$, and/or $\land$, to say that the real number $x$ is not a real number between 0 and 10, and $x$ cannot be 0 and $x$ cannot be 10.

$$\neg (x \in (0, 10))$$

Question 2:  
marks: 6

Construct a truth table for the compound proposition

$$(p \rightarrow q) \land (\neg q \rightarrow p) \rightarrow q$$

Is this proposition a tautology? Yes, No, Why?

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
<th>$\neg q \rightarrow p$</th>
<th>$(p \rightarrow q) \land (\neg q \rightarrow p)$</th>
<th>$\rightarrow q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
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</tbody>
</table>

Yes, this is a tautology.
Question 3:  

Write down the statement in predicate logic form corresponding to each of the following English sentences.

(a) I like everyone.
(b) Someone likes everyone.
(c) No one like everyone.
(d) Everyone has fun sometime.
(e) Someone has all the answers.
(f) There is no least integer.

\[ a) \quad I(x) = I \text{ like } x \]
\[ \forall x \ I(x) \]
\[ b) \quad S(x, y) = x \text{ likes } y \]
\[ \exists x \ \forall y \ S(x, y) \]
\[ c) \quad D(x, y) = x \text{ does not like } y \]
\[ \forall x \ \exists y \ D(x, y) \]
\[ d) \quad F(x, y) = x \text{ has a fun at time } y \]
\[ \forall x \ \exists y \ F(x, y) \]
\[ e) \quad A(x, y) = x \text{ has the answer to } y \]
\[ \exists x \ \forall y A(x, y) \]
\[ f) \quad L(x, y) = x < y \]
\[ \exists x \ \forall y L(x, y) \]
Question 4

$f: \mathbb{R} \rightarrow \mathbb{Z}$ be a function defined as $f(x) = \lfloor (x-1)/2 \rfloor$

a. Draw the graph of the function $f$.

b. Is $f$ one-to-one? Why?

c. Is $f$ onto? Why?

\begin{itemize}
  \item[\textcircled{a}] No, because $f(1) = 0$ and $f(2) = 0$
  \item[\textcircled{b}] Yes, because all values of the y-axis is covered and the range is $\mathbb{Z}$.\end{itemize}
Question 5

The function $f$ from $(a, b, c, d)$ to itself is $f(a) = b$, $f(b) = a$, $f(c) = d$, $f(d) = a$.

a) Is $f$ a surjection?

b) Is $f$ an injection?

Discuss briefly.

Neither a surjection nor an injection.
because no element maps on $c$, and $f(b) = a = f(d)$

Question 6

marks: 4

a) Give an example of a function $f$ from $\mathbb{N}$ to $\mathbb{N}$ such that $f$ is one-to-one but not onto.
b) Give an example where $f$ is onto but not one-to-one.

\( f(x) = 2x \)
\( f(x) = \lfloor x/2 \rfloor \)

Question 7

marks: 2

Give a "good" big-O estimate for $(2^n + n^3)(n^3 + 3^n)$

\[
(2^n + n^3)(n^3 + 3^n) = 2^n \cdot n^3 + 2^n \cdot 3^n + n^3 \cdot n^3 + n^2 \cdot 3^n \\
= 2^n \cdot n^3 + n^5 + 3^n \cdot n^2 \\
= O(6^n) \text{ [taking the largest order]}
\]
Question 8

marks: 6

Estimate the Big O for the following:

a) \( \log (56x^7 + 6x^5 + 68x^4 + 9x^3 + x^2) \)

b) \( n! \log(n!) + 4n^2 + 26 \)

c) \( \log (2^{n \log n}) \)

\[ \begin{align*}
\text{a) } & \quad \log (56x^7 + 6x^5 + 68x^4 + 9x^3 + x^2) \\
& = O(\log (56x^7)) = O(\log x^7) = O(7 \log x) = O(\log x) \\
\text{b) } & \quad n! \log(n!) + 4n^2 + 26 \\
& = O(n! \log(n!)) = O(n! n \log n) = O(n^2 (n-1)! \log n) = O(n! n! \log n) \\
\text{c) } & \quad \log (2^{n \log n}) = O(n \log n \log 2) = O(n \log n) \\
\end{align*} \]
Design an efficient algorithm to solve the following problems. Express your algorithm in terms of basic idea, input, output, pseudo code, and big “O” complexity.

1. Find the index of the last occurrence of the smallest element in a finite unsorted list of \( n \) integers. Note that the integers in the list may not be distinct.
2. Find all the indices of the smallest element in the list described above.

\[
\text{Input: list of } n \text{ integers} \\
\text{Output: index of the last smallest value} \\
\text{Basic idea:} \\
\text{Store the index of the first element in a variable index. If check for all numbers one by one (once only) if the number is less than or equal to the value stored at the position noted in index, then index is updated and the new value stored in the index is the position of the number that was compared. In the end index will contain the required value.}
\]

\[
\text{Pseudocode:} \\
\text{Input: Array of } n \text{ numbers} \\
\text{index} = 1 \\
\text{for } i = 2 \text{ to } n \\
\text{if } A[i] \leq A[\text{index}] \text{ then } index = i
\]

2. In this part, first you should find out the minimum number. Then just maintain a list of indices that contain the value of \( \text{index} \) equal to the minimum.
Question 10

marks: 7

Analyze the following program segment using big-O notation. You must show all your steps for full credit, i.e. how many times every instruction is executed.

1. $s := 0$
2. for $i := 1$ to $n$ do
3. $x := x + s$
4. next $i$
5. for $i := 1$ to $n$ do
6. for $j := 1$ to $i$ do
7. $s := s + j^*(i-j) + j^*j + n^*n*n$
8. next $j$
9. next $i$

1. 1 step
2. $n+1$ steps
3. $n$ steps
4. $n$ steps
5. $n+1$ steps
6. $i+1$ steps ($i = 0, 1, \ldots, n$)
7. $i$ steps ($i = 0, 1, \ldots, n$)
8. $i$ steps ($i = 0, 1, \ldots, n$)
9. $n$ steps

(6) and (8) are for the loop
(7) is where some calculation is taking place.
(7) takes most of the time
(7) takes $1 + 2 + 3 + \cdots + n$

$$= \frac{n(n+1)}{2} = O(n^2) \text{ steps}$$

Thus, complexity of this algorithm is $O(n^2)$
Question 11  
marks: 5

Prove the following: For all integers \( n \), and all prime numbers \( p \), if \( n^2 \) is divisible by \( p \), then \( n \) is divisible by \( p \).

\[
\begin{align*}
  n^2 \text{ is divisible by } p & \implies n \text{ is divisible by } p \\
  \iff n \text{ is not divisible by } p & \implies n^2 \text{ is not divisible by } p \\
  \Rightarrow n \text{ is not divisible by } p & \\
  \Rightarrow n = pq + r \quad \text{(quotient remainder)} \quad (r \neq 0) \\
  \Rightarrow n^2 = (pq + r)^2 = (pq)^2 + 2pq r + r^2 \\
  = p(p^2 + 2qr) + r^2 \quad \text{(and } r \neq 0) \\
  \Rightarrow n^2 \text{ is not divisible by } p \quad \text{(proved!)}
\end{align*}
\]

Question 12  
marks: 5

Prove that for the universe of integers, \( x \) is even iff \( x^2 \) is even.

\[
\begin{align*}
  x \text{ is even } & \iff x^2 \text{ is even} \\
  x \text{ is even } \rightarrow x^2 \text{ is even} \\
  x = 2a \quad \text{(even)} & \implies x^2 = (2a)^2 = 2(2a^2) \quad \text{even} \\
  x^2 \text{ is even } \rightarrow x \text{ is even} \\
  \Rightarrow x \text{ is not even } \rightarrow x^2 \text{ is not even} \\
  \Rightarrow x \text{ is odd } \rightarrow x^2 \text{ is odd} \\
  x = 2a + 1 \quad \text{odd} & \implies x^2 = (2a+1)^2 = (2a)^2 + 2 \cdot 2a + 1^2 \\
  & = 2(2a^2 + 2a) + 1 \quad \text{odd} \quad \text{proved!}
\end{align*}
\]
Question 13

Prove that if $n$ is an integer and $n^3 + 5$ is odd then $n$ is even.

$n^3 + 5$ is odd $\implies$ $n$ is even \quad [n \in \mathbb{Z}]

$\implies$ $n$ is not even $\implies$ $n^3 + 5$ is not odd

$\equiv$ $n$ is odd $\implies$ $n^3 + 5$ is even (using contrative)

$n = (2a + 1)$

$\implies$ $n^3 = (2a + 1)^3 = (2a)^3 + 3 \cdot (2a)^2 + 3 \cdot (2a) + 1$

$n^3 + 5 = 8a^3 + 12a^2 + 6a + 1 + 5$

$= 8a^3 + 12a^2 + 6a + 6$

$= 2 \cdot (4a^3 + 6a^2 + 3a + 3)$ \quad [which \ is \ even]$

proved!
Consider the compact, cute, and elegant recursive algorithm for computing Fibonacci numbers (page 217). Note that it is just a three-line algorithm. Also consider the iterative algorithm for computing Fibonacci numbers (Page 218). Note that it is more than 12 line algorithm. Derive and calculate the time complexity of each algorithm.

If we draw the tree of the recursive algorithm, it is almost a complete tree e.g. f₅

If we make it a complete tree, we get $2^n$ nodes each node does 1 addition, so complexity, # of nodes = $2^n$

For iterative, graph is a chain, # of nodes = n, => Complexity = $O(n)$
Consider example 4 on page 322. Solve this example provided $f_1 = 1$ and $f_2 = 1$. Show all steps.

$$r = \left(1 + \sqrt{5}\right) / 2$$

$$f_n = c_1 \left(\frac{1 + \sqrt{5}}{2}\right)^n + c_2 \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

$$f_1 = c_1 \left(\frac{1 + \sqrt{5}}{2}\right) + c_2 \left(\frac{1 - \sqrt{5}}{2}\right) = 1$$

$$\Rightarrow c_1 \left(1 + \sqrt{5}\right) + c_2 \left(1 - \sqrt{5}\right) = 2$$

$$\Rightarrow c_1 = \frac{2 - c_2 \left(1 - \sqrt{5}\right)}{1 + \sqrt{5}} \quad \text{(i)}$$

$$f_2 = c_1 \left(\frac{1 + \sqrt{5}}{2}\right)^2 + c_2 \left(\frac{1 - \sqrt{5}}{2}\right)^2 = 1$$

$$\Rightarrow c_1 \left(1 + \sqrt{5}\right)^2 + c_2 \left(1 - \sqrt{5}\right)^2 = 4 \quad \text{(2^2 = 4)}$$

$$\Rightarrow \left[2 - c_2 \left(1 - \sqrt{5}\right)\right] \left[\left(1 + \sqrt{5}\right)^2\right] + c_2 \left(1 - \sqrt{5}\right)^2 = 4$$

$$\Rightarrow 2 + 2\sqrt{5} - c_2 \left(1 - \sqrt{5}\right) + c_2 \left(1 + 5 - 2\sqrt{5}\right) = 4$$

$$\Rightarrow 2 + 2\sqrt{5} + c_2 \left(10 - 2\sqrt{5}\right) = 4$$

$$\Rightarrow c_2 = \frac{2 - 2\sqrt{5}}{10 - 2\sqrt{5}} = \frac{1 - \sqrt{5}}{5 - \sqrt{5}}$$

$$\Rightarrow c_2 = \frac{1 - \sqrt{5}}{5 - \sqrt{5}}$$

$$c_1 = \frac{2}{1 + \sqrt{5}} - \frac{1 - \sqrt{5}}{5 - \sqrt{5}} = \frac{36 - 5\sqrt{5}}{2\sqrt{5}}$$

$$c_1 = \frac{2}{1 + \sqrt{5}} - \frac{3\sqrt{5} - 5\sqrt{5}}{2\sqrt{5}} = \frac{2 + 2\sqrt{5}}{5\sqrt{5} + 10}$$

$$f_n = \frac{1 + \sqrt{5}}{5 + \sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^n + \frac{1 - \sqrt{5}}{5 - \sqrt{5}} \left(\frac{1 - \sqrt{5}}{2}\right)^n$$

$$\Rightarrow f_n = \left(\frac{1 + \sqrt{5}}{2^n (5 + \sqrt{5})}\right)^n + \left(\frac{1 - \sqrt{5}}{2^n (5 - \sqrt{5})}\right)^n$$