The instructions below must be followed strictly. Failure to do so can result in serious grade loss.

⇒ DO NOT OPEN THIS EXAM UNTIL TOLD TO DO SO.
⇒ Read all questions very carefully before answering them.
⇒ Check the number of papers in the question sheet and make sure the paper is complete.
⇒ Keep your eyes on your own paper.
⇒ You may not
  • Talk to anyone once the exam begins.
  • Leave the examination room and then return.

Specific instructions:

<table>
<thead>
<tr>
<th>Open book/notes, help sheet:</th>
<th>Closed Book &amp; Notes</th>
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<tbody>
<tr>
<td>Calculator usage:</td>
<td>None</td>
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<td>Write in pen/pencil:</td>
<td>Optional</td>
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<td>Any other instruction(s):</td>
<td>no communication devices, e.g. cell phone, PDA’s etc.</td>
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Score

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<tr>
<th>Q # 1</th>
<th>Q # 2</th>
<th>Q # 3</th>
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Max

Awarded
Q # 1 (16 points)
Consider the NFA, \( M = (\{q_0, q_1, q_2, q_3, q_4\}, \{a, b\}, \delta, q_0, \{q_4\}) \)

Note: \( \epsilon \) or \( \lambda \) represent the empty string.

(a) Compute the \( \epsilon \)-Closure of each state.

\[
\begin{align*}
\text{EClose}(q_0) &= \{q_0, q_1, q_2, q_3\} \\
\text{EClose}(q_1) &= \{q_1, q_2, q_3\} \\
\text{EClose}(q_2) &= \{q_2\} \\
\text{EClose}(q_3) &= \{q_3\} \\
\text{EClose}(q_4) &= \{q_3, q_4\}
\end{align*}
\]

(b) Convert the automaton to a DFA

Let the DFA \( M' = (Q_D, \{a, b\}, \delta_D, q_D, F_D) \) be equivalent to the NFA \( M \).

then \( Q_D = \{S' : S' = EClose(S) \text{ where } S \subseteq Q\} \)

\[
q_D = Eclose(q_0)
\]

\[
F_D = \{S \in Q_D : S \cap F \neq \emptyset\}
\]

For each set \( S = \{p_1, p_2, \ldots, p_k\} \subseteq Q_D \) and \( \forall a \in \Sigma \),

\[
\delta_D(S, a) = \bigcup_{j=1}^{m} EClose(r_j) \text{ where } \{r_1, r_2, \ldots, r_m\} = \bigcup_{i=1}^{k} \delta(p_i, a)
\]
Q # 2 (15 points)
Consider the DFA $A = (\{p, q\}, \{a, b\}, \delta, p, \{q\})$

The proof that $A$ accepts strings $w$ if and only if $w$ has an odd number of $a$’s is given below.

The arguments are provided; you have to give reasons why each step is true. You should choose your reason by letter, from the following options.

A. Well known facts about how integers and/or strings work,
   e.g. the sum of an odd number and an even number is odd, or the fact about strings that if $x = yz$,
   then the number of $a$’s in $x$ equals the sum of the number of $a$’s in $y$ and the number of $a$’s in $z$.
   These are not the only possible facts represented by A, but just some examples.

B. By definition of $\delta^*$.

C. By the inductive hypothesis.

Indicate option A, B, or C for each of (a) – (f) below. Then do (g)

Proof: The proof is an induction on $n$, the length of $w$, that $\delta^*(p, w) = q$ if and only if $w$ has an odd number of $a$’s.

Basis: If $n = 0$, then $w = \epsilon$. $\delta^*(p, w) = p$ and $w$ has zero $a$’s, so the statement holds.

Induction: Assume that the statement holds for $n-1$, and let $w$ be of length $n$. There are two cases: $w = xa$ or $w = xb$, where $x$ represents the first $n-1$ positions of $w$. Consider the first case, that $w = xa$. If $w$ has an even number of $a$’s then $x$ has an odd number of $a$’s because

(a) __________________

Thus $\delta^*(p, x) = q$ because

(b) __________________

Therefore $\delta^*(p, w) = p$ because

(c) __________________

Now suppose $w$ has an odd number of $a$’s then $x$ has an even number of $a$’s because

(d) __________________

$\delta^*(p, x) = p$ because

(e) __________________

$\delta^*(p, w) = q$ because

(f) __________________

These statements complete the proof of the case $w = xa$.

(g) Now give the proof of the case $w = xb$ in your own words. You may continue to use options A, B, or C as shorthand for the reasons.

Consider the case $w = xb$. If $w$ has an even number of $a$’s the $x$ has an even number of $a$’s because (A).

Thus, $\delta^*(p, x) = p$ because (C).

Therefore $\delta^*(p, w) = p$ because (B).

Now suppose $w$ has an odd number of $a$’s then $x$ has an odd number of $a$’s because (C).

$\delta^*(p, w) = q$ because (B).

These statements complete the proof of the case $w = xb$. 
Q # 3 (15 points)
The DFA from Q # 2 with states relabeled to 1 and 2 from p and q, respectively, is given below

Let \( R_{ij}^{(k)} \) be the regex that represent the set of strings \( w \) such that \( w \) is the label of a path from state \( i \) to \( j \) in DFA, and that path has no intermediate state who’s label is greater than \( k \).

Note: beginning & end points, i.e. \( i \) & \( j \) are not intermediate so they can be greater than \( k \)

What are the following expressions?

Solution:
The easiest thing to do was to understand the notation and reason out the expressions by inspection rather than using the recursive formula.

\[ R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} (R_{kk}^{(k-1)}) * R_{kj}^{(k-1)} \]

(a) \( R_{11}^{(0)} = \lambda + b \)

(b) \( R_{22}^{(1)} = R_{22}^{(0)} + R_{21}^{(0)} (R_{11}^{(0)}) * R_{12}^{(0)} \)
\[ = \lambda + b + ab* a \]

(c) \( R_{12}^{(2)} = R_{12}^{(1)} + R_{12}^{(1)} (R_{22}^{(1)}) * R_{22}^{(1)} \)
\[ = b * ab* (ab* ab*) * \]
\[ = b * a (b + ab* a) * \]

where \( R_{12}^{(1)} = R_{12}^{(0)} + R_{11}^{(0)} (R_{11}^{(0)}) * R_{12}^{(0)} \)
\[ = a + (\lambda + b)(\lambda + b) * a \]
\[ = b * a \]
Q # 4 (15 points)
Consider the DFA below $D = (\{q_0, q_1, q_2, q_3, q_4, q_5\}, \{a, b\}, \delta, q_0, \{q_4, q_5\})$

(a) Give the equivalence classes for the states of the DFA. Use the table given below for the table filling algorithm.

Equivalence Classes are
\{q_0\}, \{q_1, q_2\}, \{q_3\}, \{q_4, q_5\}

(b) Draw the minimal DFA based on the equivalence classes identified in part (a).
Q # 5 (15 points)
(a) Specify the CFG, $G$, for the language $L(G) = \{a^n b a^n : n \geq 1\}$

$G = (\{S, C\}, \{a, b\}, S, P)$

where $P = \{ S \rightarrow aCa, \ C \rightarrow aCa, \ C \rightarrow b \}$

(b) Give the derivation for $a^2 b a^2$

$S \Rightarrow aCa \Rightarrow aaCaa \Rightarrow aabaa$

(c) If the above language is modified such that $L(G') = \{a^n b a^m : n, m \geq 1\}$

Specify the CFG, $G'$

$G' = (\{S, A, B, C\}, \{a, b\}, S, P)$

where $P = \{ S \rightarrow aS, \ S \rightarrow aB, \ B \rightarrow bC, \ C \rightarrow aC, \ C \rightarrow a \}$
Q # 6 (24 points)
State whether True or False.
To get any credit prove or disprove each statement with an example or counter example.

(a) Every subset of a regular language is regular – FALSE

Counter Example
Let \( L_1 = a^* \) (regular) and \( L_2 = \{a^p : p \text{ is prime}\} \)
\( L_2 \) is not regular although \( L_2 \subset L_1 \)

(b) Let \( L_4 = L_1 L_2 L_3 \)
If \( L_1 \) and \( L_2 \) are regular and \( L_3 \) is not regular, it is possible that \( L_4 \) is regular. – TRUE

Example
Let  
\[ L_1 = \{\lambda\} \]
\[ L_2 = a^* \]
\[ L_3 = \{a^p : p \text{ is prime}\} \]
Then 
\[ L_4 = \{a^k : k \geq 2\} \] is regular as \( R = a a^* \)

(c) Regular languages, when expressed as a grammar, can only be generated by a regular (left linear or right linear) grammar. – FALSE

Counter Example
Let \( G = (\{S\}, \{a\}, S, P) \)
where \( P = \{ S \rightarrow SS | a | \lambda \} \)
\( G \) not a linear grammar but \( L(G) = a^* \) is regular