3.2. FORMAL DEFINITION OF LIMITS

3.2.2 Example

What if we want the function to be within $\epsilon$ of 6?

\[ |2x + 2 - 6| < \epsilon \]
\[ |2x - 4| < \epsilon \]
\[ |2(x - 2)| < \epsilon \]
\[ |x - 2| < \epsilon/2 \]

Now, if we read the above in reverse we would see that

\[ |x - 2| < \delta = \epsilon/2 \implies |2x + 2 - 6| < \epsilon \]

So, no matter how small the tolerance $\epsilon$ gets, we can make the function within $\epsilon$ of 6 by taking $x$ to within $\epsilon/2$ of 2.

3.2.3 Formal definition of limits

The limit of $f(x)$ as $x$ approaches $x_0$ is $L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ depending on $\epsilon$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - x_0| < \delta$.

3.2.4 Example

We’ve shown previously that

\[ \lim_{x \to 2} 2x + 2 = 6 \]

3.2.5 Exercise

Find an interval $(a, b)$ that contains $x = 2$ so that when $x$ is in the interval $|2x + 2 - 6| < 0.1$. 
Chapter 4

One-sided limits

4.1 One-sided limits

4.1.1 Idea

In the expression

$$\lim_{x \to x_0} f(x)$$

we consider how the function $f$ behaves as we approach $x$, both from the left and from the right on the real line. But, sometimes we need to look at what happens in a particular direction. The limit may have one value in one direction and another value in the other direction. The limit may not exist in one direction but exist in the other. The one-sided limits may be the same in both directions, in which case the limit exists.

4.1.2 Right-handed limit

We write

$$\lim_{x \to x_0^+} f(x) = L$$

if as $x$ becomes arbitrarily close to $x_0$, but never equal to or less than $x_0$, $f(x)$ becomes arbitrarily close to $L$.

4.1.3 Left-handed limit

We write
CHAPTER 4. ONE-SIDED LIMITS

\[ \lim_{x \to x_0^-} f(x) = L \]

if as \( x \) becomes arbitrarily close to \( x_0 \), but never equal to or greater than \( x_0 \), \( f(x) \) becomes arbitrarily close to \( L \).

4.1.4 Examples

Example

Consider a step function \( H(x) \) which is zero for \( x \) negative and one for \( x \) non-negative. Draw the graph. Find

\[ \lim_{x \to 0^+} H(x) \]

\[ \lim_{x \to 0^-} H(x) \]

Note that because these two results are different the regular limit

\[ \lim_{x \to 0} H(x) \]

does not exist.

Example

Let \( f(x) = \frac{1}{x} \). Graph \( f(x) \). Note that the right-handed limit at zero does not exist and the left-handed limit at zero does not exist.

Example

Consider the function \( y(x) \) whose graph is the top half of the unit circle \( x^2 + y^2 = 1 \). Draw the graph and answer the following:

Why do the following limits not exist?

\[ \lim_{x \to 1^+} y(x) \]

\[ \lim_{x \to 1^-} y(x) \]

What are the following limits?

\[ \lim_{x \to 1^-} y(x) \]

\[ \lim_{x \to 1^+} y(x) \]
4.2 One-sided limits and regular limits

If
\[ \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L \]
then the regular limit exists and
\[ \lim_{x \to x_0} f(x) = L \]
On the other hand, if
\[ \lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x) \]
then the regular limit does not exist.

4.2.1 Exercises

Exercise

Draw the graphs of: a) A function whose right-handed limit exists and left-handed limit does not exist; b) A function whose left-handed limit exists and whose right-handed limit exists; c) A function whose one-sided limits exist and are different.

Exercise

Why does the limit
\[ \lim_{x \to 0} \frac{x}{|x|} \]
not exist?

4.2.2 Formal definition of one-sided limits

Formal definition of right-handed limits

We write
\[ \lim_{x \to x_0^+} f(x) = L \]
if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that
\[ x_0 < x < x_0 + \delta \implies |f(x) - L| < \epsilon \]
Formal definition of left-handed limits

We write

\[ \lim_{x \to x_0^-} f(x) = L \]

if for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that

\[ x_0 - \delta < x < x_0 \implies |f(x) - L| < \epsilon \]